6.6 POISSON'S AND LAPLACE'S EQUATIONS

In preceding sections, we have found capacitance by first assuming a known charge distribution on the conductors and then finding the potential difference in terms of the assumed charge. An alternate approach would be to start with known potentials on each conductor, and then work backward to find the charge in terms of the known potential difference. The capacitance in either case is found by the ratio Q/V.

The first objective in the latter approach is thus to find the potential function between conductors, given values of potential on the boundaries, along with possible volume charge densities in the region of interest. The mathematical tools that enable this to happen are Poisson's and Laplace's equations, to be explored in the remainder of this chapter. Problems involving one to three dimensions can be solved either analytically or numerically. Laplace's and Poisson's equations, when compared to other methods, are probably the most widely useful because many problems in engineering practice involve devices in which applied potential differences are known, and in which constant potentials occur at the boundaries.

Obtaining Poisson's equation is exceedingly simple, for from the point form of Gauss's law,

$$\nabla \cdot \mathbf{D} = \rho_\nu \tag{21}$$

the definition of **D**,

$$\mathbf{D} = \epsilon \mathbf{E} \tag{22}$$

and the gradient relationship,

$$\mathbf{E} = -\nabla V \tag{23}$$

by substitution we have

$$\nabla \cdot \mathbf{D} = \nabla \cdot (\epsilon \mathbf{E}) = -\nabla \cdot (\epsilon \nabla V) = \rho_\nu$$

or

$$\nabla \cdot \nabla V = -\frac{\rho_\nu}{\epsilon} \tag{24}$$

for a homogeneous region in which ϵ is constant.

Equation (24) is *Poisson's equation,* but the "double ∇" operation must be interpreted and expanded, at least in rectangular coordinates, before the equation can be useful. In rectangular coordinates,

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z$$

and therefore

$$\nabla \cdot \nabla V = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial z}\right)$$
$$= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \tag{25}$$

Usually the operation $\nabla \cdot \nabla$ is abbreviated ∇^2 (and pronounced "del squared"), a good reminder of the second-order partial derivatives appearing in Eq. (5), and we have

$$\boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\epsilon}} \tag{26}$$

in rectangular coordinates.

If $\rho_v = 0$, indicating zero *volume* charge density, but allowing point charges, line charge, and surface charge density to exist at singular locations as sources of the field, then

$$\boxed{\nabla^2 V = 0} \tag{27}$$

which is *Laplace's equation*. The ∇^2 operation is called the *Laplacian of V*.

In rectangular coordinates Laplace's equation is

$$\boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad \text{(rectangular)}} \tag{28}$$

and the form of $\nabla^2 V$ in cylindrical and spherical coordinates may be obtained by using the expressions for the divergence and gradient already obtained in those coordinate systems. For reference, the Laplacian in cylindrical coordinates is

$$\boxed{\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2} \qquad \text{(cylindrical)}} \tag{29}$$

and in spherical coordinates is

$$\boxed{\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} \qquad \text{(spherical)}} \tag{30}$$

These equations may be expanded by taking the indicated partial derivatives, but it is usually more helpful to have them in the forms just given; furthermore, it is much easier to expand them later if necessary than it is to put the broken pieces back together again.

Laplace's equation is all-embracing, for, applying as it does wherever volume charge density is zero, it states that every conceivable configuration of electrodes or conductors produces a field for which $\nabla^2 V = 0$. All these fields are different, with different potential values and different spatial rates of change, yet for each of them $\nabla^2 V = 0$. Because *every* field (if $\rho_v = 0$) satisfies Laplace's equation, how can we expect to reverse the procedure and use Laplace's equation to find one specific field in which we happen to have an interest? Obviously, more information is required, and we shall find that we must solve Laplace's equation subject to certain *boundary conditions*.

Every physical problem must contain at least one conducting boundary and usually contains two or more. The potentials on these boundaries are assigned values, perhaps $V_0, V_1, \ldots$, or perhaps numerical values. These definite equipotential surfaces will provide the boundary conditions for the type of problem to be solved. In other types of problems, the boundary conditions take the form of specified values of E (alternatively, a surface charge density, ρ_S) on an enclosing surface, or a mixture of known values of V and E.

Before using Laplace's equation or Poisson's equation in several examples, we must state that if our answer satisfies Laplace's equation and also satisfies the boundary conditions, then it is the only possible answer. This is a statement of the Uniqueness Theorem, the proof of which is presented in Appendix D.

D6.5. Calculate numerical values for V and ρ_v at point P in free space if: (a) $V = \dfrac{4yz}{x^2+1}$, at $P(1, 2, 3)$; (b) $V = 5\rho^2 \cos 2\phi$, at $P(\rho = 3, \phi = \dfrac{\pi}{3}$, $z = 2)$; (c) $V = \dfrac{2\cos\phi}{r^2}$, at $P(r = 0.5, \theta = 45^\circ, \phi = 60^\circ)$.

Ans. 12 V, -106.2 pC/m^3; -22.5 V, 0; 4 V, 0

6.7 EXAMPLES OF THE SOLUTION OF LAPLACE'S EQUATION

Several methods have been developed for solving Laplace's equation. The simplest method is that of direct integration. We will use this technique to work several examples involving one-dimensional potential variation in various coordinate systems in this section.

The method of direct integration is applicable only to problems that are "one-dimensional," or in which the potential field is a function of only one of the three coordinates. Since we are working with only three coordinate systems, it might seem, then, that there are nine problems to be solved, but a little reflection will show that a field that varies only with x is fundamentally the same as a field that varies only with y. Rotating the physical problem a quarter turn is no change. Actually, there are only five problems to be solved, one in rectangular coordinates, two in cylindrical, and two in spherical. We will solve them all.

First, let us assume that V is a function only of x and worry later about which physical problem we are solving when we have a need for boundary conditions. Laplace's equation reduces to

$$\frac{\partial^2 V}{\partial x^2} = 0$$

and the partial derivative may be replaced by an ordinary derivative, since V is not a function of y or z,

$$\frac{d^2V}{dx^2} = 0$$

We integrate twice, obtaining

$$\frac{dV}{dx} = A$$

and

$$V = Ax + B \tag{31}$$

where A and B are constants of integration. Equation (31) contains two such constants, as we would expect for a second-order differential equation. These constants can be determined only from the boundary conditions.

Since the field varies only with x and is not a function of y and z, then V is a constant if x is a constant or, in other words, the equipotential surfaces are parallel planes normal to the x axis. The field is thus that of a parallel-plate capacitor, and as soon as we specify the potential on any two planes, we may evaluate our constants of integration.

EXAMPLE 6.2

Start with the potential function, Eq. (31), and find the capacitance of a parallel-plate capacitor of plate area S, plate separation d, and potential difference V_0 between plates.

Solution. Take $V = 0$ at $x = 0$ and $V = V_0$ at $x = d$. Then from Eq. (31),

$$A = \frac{V_0}{d} \qquad B = 0$$

and

$$\boxed{V = \frac{V_0 x}{d}} \tag{32}$$

We still need the total charge on either plate before the capacitance can be found. We should remember that when we first solved this capacitor problem, the sheet of charge provided our starting point. We did not have to work very hard to find the charge, for all the fields were expressed in terms of it. The work then was spent in finding potential difference. Now the problem is reversed (and simplified).

The necessary steps are these, after the choice of boundary conditions has been made:

1. Given V, use $\mathbf{E} = -\nabla V$ to find $\mathbf{E}$.
2. Use $\mathbf{D} = \epsilon\mathbf{E}$ to find $\mathbf{D}$.
3. Evaluate $\mathbf{D}$ at either capacitor plate, $\mathbf{D} = \mathbf{D}_S = D_N \mathbf{a}_N$.
4. Recognize that $\rho_S = D_N$.
5. Find Q by a surface integration over the capacitor plate, $Q = \int_S \rho_S \, dS$.

Here we have

$$V = V_0 \frac{x}{d}$$

$$\mathbf{E} = -\frac{V_0}{d}\mathbf{a}_x$$

$$\mathbf{D} = -\epsilon \frac{V_0}{d}\mathbf{a}_x$$

$$\mathbf{D}_S = \mathbf{D}\Big|_{x=0} = -\epsilon \frac{V_0}{d}\mathbf{a}_x$$

$$\mathbf{a}_N = \mathbf{a}_x$$

$$D_N = -\epsilon \frac{V_0}{d} = \rho_S$$

$$Q = \int_S \frac{-\epsilon V_0}{d} dS = -\epsilon \frac{V_0 S}{d}$$

and the capacitance is

$$\boxed{C = \frac{|Q|}{V_0} = \frac{\epsilon S}{d}} \tag{33}$$

We will use this procedure several times in the examples to follow.

EXAMPLE 6.3

Because no new problems are solved by choosing fields which vary only with y or with z in rectangular coordinates, we pass on to cylindrical coordinates for our next example. Variations with respect to z are again nothing new, and we next assume variation with respect to ρ only. Laplace's equation becomes

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) = 0$$

Noting the ρ in the denominator, we exclude $\rho = 0$ from our solution and then multiply by ρ and integrate,

$$\rho \frac{dV}{d\rho} = A$$

where a total derivative replaces the partial derivative because V varies only with ρ. Next, rearrange, and integrate again,

$$V = A \ln \rho + B \tag{34}$$

The equipotential surfaces are given by $\rho =$ constant and are cylinders, and the problem is that of the coaxial capacitor or coaxial transmission line. We choose a potential difference of V_0 by letting $V = V_0$ at $\rho = a$, $V = 0$ at $\rho = b$, $b > a$, and obtain

$$\boxed{V = V_0 \frac{\ln(b/\rho)}{\ln(b/a)}} \tag{35}$$

from which

$$\mathbf{E} = \frac{V_0}{\rho} \frac{1}{\ln(b/a)} \mathbf{a}_\rho$$

$$D_{N(\rho=a)} = \frac{\epsilon V_0}{a \ln(b/a)}$$

$$Q = \frac{\epsilon V_0 2\pi a L}{a \ln(b/a)}$$

$$\boxed{C = \frac{2\pi\epsilon L}{\ln(b/a)}} \tag{36}$$

which agrees with our result in Section 6.3 (Eq. (5)).

EXAMPLE 6.4

Now assume that V is a function only of ϕ in cylindrical coordinates. We might look at the physical problem first for a change and see that equipotential surfaces are given by $\phi =$ constant. These are radial planes. Boundary conditions might be $V = 0$ at $\phi = 0$ and $V = V_0$ at $\phi = \alpha$, leading to the physical problem detailed in Figure 6.10.

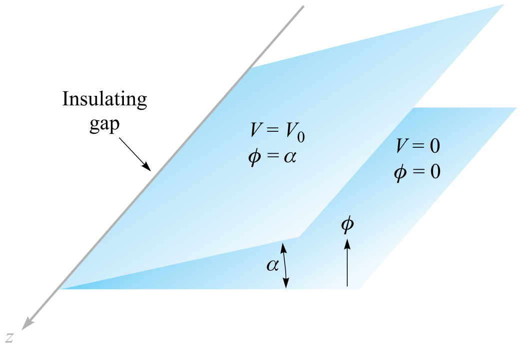

Figure 6.10 Two infinite radial planes with an interior angle α. An infinitesimal insulating gap exists at $\rho = 0$. The potential field may be found by applying Laplace's equation in cylindrical coordinates.

Laplace's equation is now

$$\frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} = 0$$

We exclude $\rho = 0$ and have

$$\frac{d^2 V}{d\phi^2} = 0$$

The solution is

$$V = A\phi + B$$

The boundary conditions determine A and B, and

$$\boxed{V = V_0 \frac{\phi}{\alpha}} \tag{37}$$

Taking the gradient of Eq. (37) produces the electric field intensity,

$$\boxed{\mathbf{E} = -\frac{V_0 \mathbf{a}_\phi}{\alpha\rho}} \tag{38}$$

and it is interesting to note that E is a function of ρ and not of ϕ. This does not contradict our original assumptions, which were restrictions only on the potential field. Note, however, that the *vector* field $\mathbf{E}$ is in the ϕ direction.

A problem involving the capacitance of these two radial planes is included at the end of the chapter.

EXAMPLE 6.5

We now turn to spherical coordinates, dispose immediately of variations with respect to ϕ only as having just been solved, and treat first $V = V(r)$.

The details are left for a problem later, but the final potential field is given by

$$\boxed{V = V_0 \frac{\dfrac{1}{r} - \dfrac{1}{b}}{\dfrac{1}{a} - \dfrac{1}{b}}} \tag{39}$$

where the boundary conditions are evidently $V = 0$ at $r = b$ and $V = V_0$ at $r = a$, $b > a$. The problem is that of concentric spheres. The capacitance was found previously in Section 6.3 (by a somewhat different method) and is

$$\boxed{C = \frac{4\pi\epsilon}{\dfrac{1}{a} - \dfrac{1}{b}}} \tag{40}$$

EXAMPLE 6.6

In spherical coordinates we now restrict the potential function to $V = V(\theta)$, obtaining

$$\frac{1}{r^2 \sin\theta} \frac{d}{d\theta}\left(\sin\theta \frac{dV}{d\theta}\right) = 0$$

We exclude $r = 0$ and $\theta = 0$ or π and have

$$\sin\theta \frac{dV}{d\theta} = A$$

The second integral is then

$$V = \int \frac{A\,d\theta}{\sin\theta} + B$$

which is not as obvious as the previous ones. From integral tables (or a good memory) we have

$$V = A \ln\left(\tan\frac{\theta}{2}\right) + B \tag{41}$$

The equipotential surfaces of Eq. (41) are cones. Figure 6.11 illustrates the case where $V = 0$ at $\theta = \pi/2$ and $V = V_0$ at $\theta = \alpha$, $\alpha < \pi/2$. We obtain

$$\boxed{V = V_0 \frac{\ln\left(\tan\frac{\theta}{2}\right)}{\ln\left(\tan\frac{\alpha}{2}\right)}} \tag{42}$$

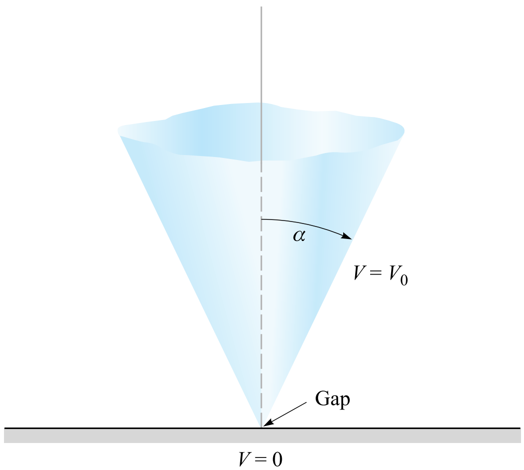

Figure 6.11 For the cone $\theta = \alpha$ at V_0 and the plane $\theta = \pi/2$ at $V = 0$, the potential field is given by $V = V_0[\ln(\tan\theta/2)]/[\ln(\tan\alpha/2)]$.

In order to find the capacitance between a conducting cone with its vertex separated from a conducting plane by an infinitesimal insulating gap and its axis normal to the plane, we first find the field strength:

$$\mathbf{E} = -\nabla V = \frac{-1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta = -\frac{V_0}{r\sin\theta \ln\left(\tan\dfrac{\alpha}{2}\right)}\mathbf{a}_\theta$$

The surface charge density on the cone is then

$$\rho_S = \frac{-\epsilon V_0}{r\sin\alpha \ln\left(\tan\dfrac{\alpha}{2}\right)}$$

producing a total charge Q,

$$Q = \frac{-\epsilon V_0}{\sin\alpha \ln\left(\tan\dfrac{\alpha}{2}\right)} \int_0^\infty \int_0^{2\pi} \frac{r\sin\alpha\, d\phi\, dr}{r}$$

$$= \frac{-2\pi\epsilon_0 V_0}{\ln\left(\tan\dfrac{\alpha}{2}\right)} \int_0^\infty dr$$

This leads to an infinite value of charge and capacitance, and it becomes necessary to consider a cone of finite size. Our answer will now be only an approximation because the theoretical equipotential surface is $\theta = \alpha$, a conical surface extending from $r = 0$ to $r = \infty$, whereas our physical conical surface extends only from $r = 0$ to, say, $r = r_1$. The approximate capacitance is

$$\boxed{C \doteq \frac{2\pi\epsilon r_1}{\ln\left(\cot\dfrac{\alpha}{2}\right)}} \tag{43}$$

If we desire a more accurate answer, we may make an estimate of the capacitance of the base of the cone to the zero-potential plane and add this amount to our answer. Fringing, or nonuniform, fields in this region have been neglected and introduce an additional source of error.

D6.6. Find $|\mathbf{E}|$ at $P(3, 1, 2)$ in rectangular coordinates for the field of: (*a*) two coaxial conducting cylinders, $V = 50$ V at $\rho = 2$ m, and $V = 20$ V at $\rho = 3$ m; (*b*) two radial conducting planes, $V = 50$ V at $\phi = 10^\circ$, and $V = 20$ V at $\phi = 30^\circ$.

Ans. 23.4 V/m; 27.2 V/m

6.8 EXAMPLE OF THE SOLUTION OF POISSON'S EQUATION: THE P-N JUNCTION CAPACITANCE

To select a reasonably simple problem that might illustrate the application of Poisson's equation, we must assume that the volume charge density is specified. This is not usually the case, however; in fact, it is often the quantity about which we are seeking further information. The type of problem which we might encounter later would begin with a knowledge only of the boundary values of the potential, the electric field intensity, and the current density. From these we would have to apply Poisson's equation, the continuity equation, and some relationship expressing the forces on the charged particles, such as the Lorentz force equation or the diffusion equation, and solve the whole system of equations simultaneously. Such an ordeal is beyond the scope of this text, and we will therefore assume a reasonably large amount of information.

As an example, let us select a *pn* junction between two halves of a semiconductor bar extending in the x direction. We will assume that the region for $x < 0$ is doped p type and that the region for $x > 0$ is n type. The degree of doping is identical on each side of the junction. To review some of the facts about the semiconductor junction, we note that initially there are excess holes to the left of the junction and excess electrons to the right. Each diffuses across the junction until an electric field is built up in such a direction that the diffusion current drops to zero. Thus, to prevent more holes from moving to the right, the electric field in the neighborhood of the junction must be directed to the left; E_x is negative there. This field must be produced by a net positive charge to the right of the junction and a net negative charge to the left. Note that the layer of positive charge consists of two parts—the holes which have crossed the junction and the positive donor ions from which the electrons have departed. The negative layer of charge is constituted in the opposite manner by electrons and negative acceptor ions.

The type of charge distribution that results is shown in Figure 6.12*a*, and the negative field which it produces is shown in Figure 6.12*b*. After looking at these two figures, one might profitably read the previous paragraph again.

A charge distribution of this form may be approximated by many different expressions. One of the simpler expressions is

$$\rho_v = 2\rho_{v0} \operatorname{sech} \frac{x}{a} \tanh \frac{x}{a} \tag{44}$$

which has a maximum charge density $\rho_{v,max} = \rho_{v0}$ that occurs at $x = 0.881a$. The maximum charge density ρ_{v0} is related to the acceptor and donor concentrations N_a and N_d by noting that all the donor and acceptor ions in this region (the *depletion* layer) have been stripped of an electron or a hole, and thus

$$\rho_{v0} = eN_a = eN_d$$

We now solve Poisson's equation,

$$\nabla^2 V = -\frac{\rho_v}{\epsilon}$$

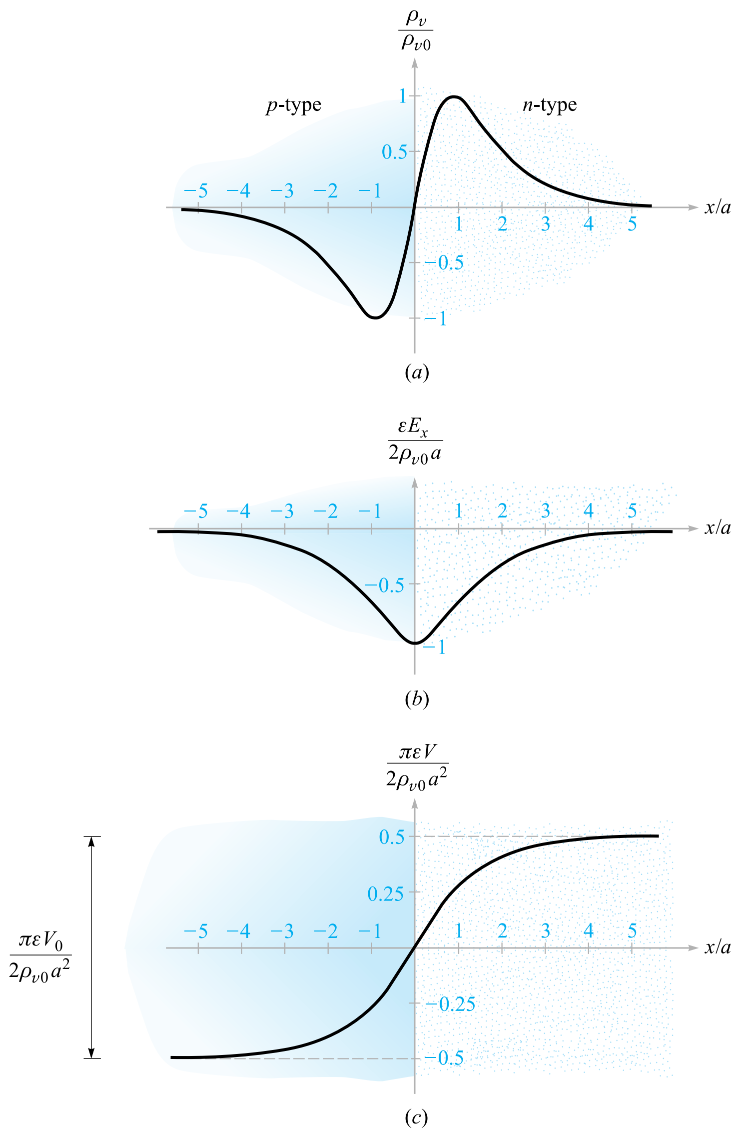

Figure 6.12 (*a*) The charge density, (*b*) the electric field intensity, and (*c*) the potential are plotted for a *pn* junction as functions of distance from the center of the junction. The *p*-type material is on the left, and the *n*-type is on the right.

subject to the charge distribution assumed above,

$$\frac{d^2V}{dx^2} = -\frac{2\rho_{v0}}{\epsilon}\operatorname{sech}\frac{x}{a}\tanh\frac{x}{a}$$

in this one-dimensional problem in which variations with y and z are not present. We integrate once,

$$\frac{dV}{dx} = \frac{2\rho_{v0}a}{\epsilon}\operatorname{sech}\frac{x}{a} + C_1$$

and obtain the electric field intensity,

$$E_x = -\frac{2\rho_{v0}a}{\epsilon}\operatorname{sech}\frac{x}{a} - C_1$$

To evaluate the constant of integration C_1, we note that no net charge density and no fields can exist *far* from the junction. Thus, as $x \to \pm\infty$, E_x must approach zero. Therefore $C_1 = 0$, and

$$E_x = -\frac{2\rho_{v0}a}{\epsilon}\operatorname{sech}\frac{x}{a} \tag{45}$$

Integrating again,

$$V = \frac{4\rho_{v0}a^2}{\epsilon}\tan^{-1}e^{x/a} + C_2$$

Let us arbitrarily select our zero reference of potential at the center of the junction, $x = 0$,

$$0 = \frac{4\rho_{v0}a^2}{\epsilon}\frac{\pi}{4} + C_2$$

and finally,

$$V = \frac{4\rho_{v0}a^2}{\epsilon}\left(\tan^{-1}e^{x/a} - \frac{\pi}{4}\right) \tag{46}$$

Figure 6.12 shows the charge distribution (a), electric field intensity (b), and the potential (c), as given by Eqs. (44), (45), and (46), respectively.

The potential is constant once we are a distance of about $4a$ or $5a$ from the junction. The total potential difference V_0 across the junction is obtained from Eq. (46),

$$V_0 = V_{x\to\infty} - V_{x\to-\infty} = \frac{2\pi\rho_{v0}a^2}{\epsilon} \tag{47}$$

This expression suggests the possibility of determining the total charge on one side of the junction and then using Eq. (47) to find a junction capacitance. The total positive charge is

$$Q = S\int_0^\infty 2\rho_{v0}\operatorname{sech}\frac{x}{a}\tanh\frac{x}{a}\,dx = 2\rho_{v0}aS$$

where S is the area of the junction cross section. If we make use of Eq. (47) to eliminate the distance parameter a, the charge becomes

$$Q = S\sqrt{\frac{2\rho_{v0}\epsilon V_0}{\pi}} \tag{48}$$

Because the total charge is a function of the potential difference, we have to be careful in defining a capacitance. Thinking in "circuit" terms for a moment,

$$I = \frac{dQ}{dt} = C\frac{dV_0}{dt}$$

and thus

$$C = \frac{dQ}{dV_0}$$

By differentiating Eq. (48), we therefore have the capacitance

$$C = \sqrt{\frac{\rho_{\nu 0}\epsilon}{2\pi V_0}}S = \frac{\epsilon S}{2\pi a} \tag{49}$$

The first form of Eq. (49) shows that the capacitance varies inversely as the square root of the voltage. That is, a higher voltage causes a greater separation of the charge layers and a smaller capacitance. The second form is interesting in that it indicates that we may think of the junction as a parallel-plate capacitor with a "plate" separation of $2\pi a$. In view of the dimensions of the region in which the charge is concentrated, this is a logical result.

Poisson's equation enters into any problem involving volume charge density. Besides semiconductor diode and transistor models, we find that vacuum tubes, magnetohydrodynamic energy conversion, and ion propulsion require its use in constructing satisfactory theories.

D6.7. In the neighborhood of a certain semiconductor junction, the volume charge density is given by $\rho_\nu = 750 \text{ sech } 10^6\pi x \tanh 10^6\pi x$ C/m^3. The dielectric constant of the semiconductor material is 10 and the junction area is 2×10^{-7} m^2. Find: (*a*) V_0; (*b*) C; (*c*) E at the junction.

Ans. 2.70 V; 8.85 pF; 2.70 MV/m

D6.8. Given the volume charge density $\rho_\nu = -2 \times 10^7 \epsilon_0 \sqrt{x}$ C/m^3 in free space, let $V = 0$ at $x = 0$ and let $V = 2$ V at $x = 2.5$ mm. At $x = 1$ mm, find: (*a*) V; (*b*) E_x.

Ans. 0.302 V; −555 V/m

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CHAPTER 6 PROBLEMS

WW Quizzes

6.1 Consider a coaxial capacitor having inner radius a, outer radius b, unit length, and filled with a material with dielectric constant, ϵ_r. Compare this to a parallel-plate capacitor having plate width w, plate separation d, filled with the same dielectric, and having unit length. Express the ratio b/a in terms of the ratio d/w, such that the two structures will store the same energy for a given applied voltage.

6.2 Let $S = 100\ \text{mm}^2$, $d = 3$ mm, and $\epsilon_r = 12$ for a parallel-plate capacitor. (*a*) Calculate the capacitance. (*b*) After connecting a 6-V battery across the capacitor, calculate E, D, Q, and the total stored electrostatic energy. (*c*) With the source still connected, the dielectric is carefully withdrawn from between the plates. With the dielectric gone, recalculate E, D, Q, and the energy stored in the capacitor. (*d*) If the charge and energy found in part (*c*) are less than the values found in part (*b*) (which you should have discovered), what became of the missing charge and energy?

6.3 Capacitors tend to be more expensive as their capacitance and maximum voltage V_{max} increase. The voltage V_{max} is limited by the field strength at which the dielectric breaks down, E_{BD}. Which of these dielectrics will give the largest CV_{max} product for equal plate areas? (*a*) Air: $\epsilon_r = 1$, $E_{BD} = 3$ MV/m. (*b*) Barium titanate: $\epsilon_r = 1200$, $E_{BD} = 3$ MV/m. (*c*) Silicon dioxide: $\epsilon_r = 3.78$, $E_{BD} = 16$ MV/m. (*d*) Polyethylene: $\epsilon_r = 2.26$, $E_{BD} = 4.7$ MV/m.

6.4 An air-filled parallel-plate capacitor with plate separation d and plate area A is connected to a battery that applies a voltage V_0 between plates. With the battery left connected, the plates are moved apart to a distance of $10d$. Determine by what factor each of the following quantities changes: (*a*) V_0; (*b*) C; (*c*) E; (*d*) D; (*e*) Q; (*f*) ρ_S; (*g*) W_E.

6.5 A parallel-plate capacitor is filled with a nonuniform dielectric characterized by $\epsilon_r = 2 + 2 \times 10^6 x^2$, where x is the distance from one plate in meters. If $S = 0.02\ \text{m}^2$ and $d = 1$ mm, find C.

6.6 Repeat Problem 6.4, assuming the battery is disconnected before the plate separation is increased.

6.7 Let $\epsilon_{r1} = 2.5$ for $0 < y < 1$ mm, $\epsilon_{r2} = 4$ for $1 < y < 3$ mm, and ϵ_{r3} for $3 < y < 5$ mm (region 3). Conducting surfaces are present at $y = 0$ and

$y = 5$ mm. Calculate the capacitance per square meter of surface area if (*a*) region 3 is air; (*b*) $\epsilon_{r3} = \epsilon_{r1}$; (*c*) $\epsilon_{r3} = \epsilon_{r2}$; (*d*) region 3 is silver.

6.8 A parallel-plate capacitor is made using two circular plates of radius a, with the bottom plate on the xy plane, centered at the origin. The top plate is located at $z = d$, with its center on the z axis. Potential V_0 is on the top plate; the bottom plate is grounded. Dielectric having *radially dependent* permittivity fills the region between plates. The permittivity is given by $\epsilon(\rho) = \epsilon_0(1 + \rho^2/a^2)$. Find (*a*) **E**; (*b*) **D**; (*c*) Q; (*d*) C.

6.9 Two coaxial conducting cylinders of radius 2 cm and 4 cm have a length of 1 m. The region between the cylinders contains a layer of dielectric from $\rho = c$ to $\rho = d$ with $\epsilon_r = 4$. Find the capacitance if (*a*) $c = 2$ cm, $d = 3$ cm; (*b*) $d = 4$ cm, and the volume of the dielectric is the same as in part (*a*).

6.10 A coaxial cable has conductor dimensions of $a = 1.0$ mm and $b = 2.7$ mm. The inner conductor is supported by dielectric spacers ($\epsilon_r = 5$) in the form of washers with a hole radius of 1 mm and an outer radius of 2.7 mm, and with a thickness of 3.0 mm. The spacers are located every 2 cm down the cable. (*a*) By what factor do the spacers increase the capacitance per unit length? (*b*) If 100 V is maintained across the cable, find **E** at all points.

6.11 Two conducting spherical shells have radii $a = 3$ cm and $b = 6$ cm. The interior is a perfect dielectric for which $\epsilon_r = 8$. (*a*) Find C. (*b*) A portion of the dielectric is now removed so that $\epsilon_r = 1.0$, $0 < \phi < \pi/2$, and $\epsilon_r = 8$, $\pi/2 < \phi < 2\pi$. Again find C.

6.12 (*a*) Determine the capacitance of an isolated conducting sphere of radius a in free space (consider an outer conductor existing at $r \to \infty$). (*b*) The sphere is to be covered with a dielectric layer of thickness d and dielectric contant ϵ_r. If $\epsilon_r = 3$, find d in terms of a such that the capacitance is twice that of part (*a*).

6.13 With reference to Figure 6.5, let $b = 6$ m, $h = 15$ m, and the conductor potential be 250 V. Take $\epsilon = \epsilon_0$. Find values for K_1, ρ_L, a, and C.

6.14 Two #16 copper conductors (1.29 mm diameter) are parallel with a separation d between axes. Determine d so that the capacitance between wires in air is 30 pF/m.

6.15 A 2-cm-diameter conductor is suspended in air with its axis 5 cm from a conducting plane. Let the potential of the cylinder be 100 V and that of the plane be 0 V. (*a*) Find the surface charge density on the cylinder at a point nearest the plane. (*b*) Plane at a point nearest the cylinder; (*c*) find the capacitance per unit length.

6.16 Consider an arrangement of two isolated conducting surfaces of any shape that form a capacitor. Use the definitions of capacitance (Eq. (2) in this chapter) and resistance (Eq. (14) in Chapter 5) to show that when the region between the conductors is filled with either conductive material (conductivity σ) or a perfect dielectric (permittivity ϵ), the resulting resistance and capacitance of the structures are related through the simple formula $RC = \epsilon/\sigma$. What basic properties must be true about both the dielectric and the conducting medium for this condition to hold for certain?

6.17 Construct a curvilinear-square map for a coaxial capacitor of 3 cm inner radius and 8 cm outer radius. These dimensions are suitable for the drawing. (*a*) Use your sketch to calculate the capacitance per meter length, assuming $\epsilon_r = 1$. (*b*) Calculate an exact value for the capacitance per unit length.

6.18 Construct a curvilinear-square map of the potential field about two parallel circular cylinders, each of 2.5 cm radius, separated by a center-to-center distance of 13 cm. These dimensions are suitable for the actual sketch if symmetry is considered. As a check, compute the capacitance per meter both from your sketch and from the exact formula. Assume $\epsilon_r = 1$.

6.19 Construct a curvilinear-square map of the potential field between two parallel circular cylinders, one of 4 cm radius inside another of 8 cm radius. The two axes are displaced by 2.5 cm. These dimensions are suitable for the drawing. As a check on the accuracy, compute the capacitance per meter from the sketch and from the exact expression:

$$C = \frac{2\pi\epsilon}{\cosh^{-1}\left[(a^2 + b^2 - D^2)/(2ab)\right]}$$

where a and b are the conductor radii and D is the axis separation.

6.20 A solid conducting cylinder of 4 cm radius is centered within a rectangular conducting cylinder with a 12 cm by 20 cm cross section. (*a*) Make a full-size sketch of one quadrant of this configuration and construct a curvilinear-square map for its interior. (*b*) Assume $\epsilon = \epsilon_0$ and estimate C per meter length.

6.21 The inner conductor of the transmission line shown in Figure 6.13 has a square cross section $2a \times 2a$, whereas the outer square is $4a \times 5a$. The axes are displaced as shown. (*a*) Construct a good-sized drawing of this transmission line, say with $a = 2.5$ cm, and then prepare a curvilinear-square plot of the electrostatic field between the conductors. (*b*) Use the map to calculate the capacitance per meter length if $\epsilon = 1.6\epsilon_0$. (*c*) How would your result to part (*b*) change if $a = 0.6$ cm?

6.22 Two conducting plates, each 3×6 cm, and three slabs of dielectric, each $1 \times 3 \times 6$ cm, and having dielectric constants of 1, 2, and 3, are assembled into a capacitor with $d = 3$ cm. Determine the two values of capacitance obtained by the two possible methods of assembling the capacitor.

6.23 A two-wire transmission line consists of two parallel perfectly conducting cylinders, each having a radius of 0.2 mm, separated by a center-to-center distance of 2 mm. The medium surrounding the wires has $\epsilon_r = 3$ and $\sigma = 1.5$ mS/m. A 100-V battery is connected between the wires. (*a*) Calculate the magnitude of the charge per meter length on each wire. (*b*) Using the result of Problem 6.16, find the battery current.

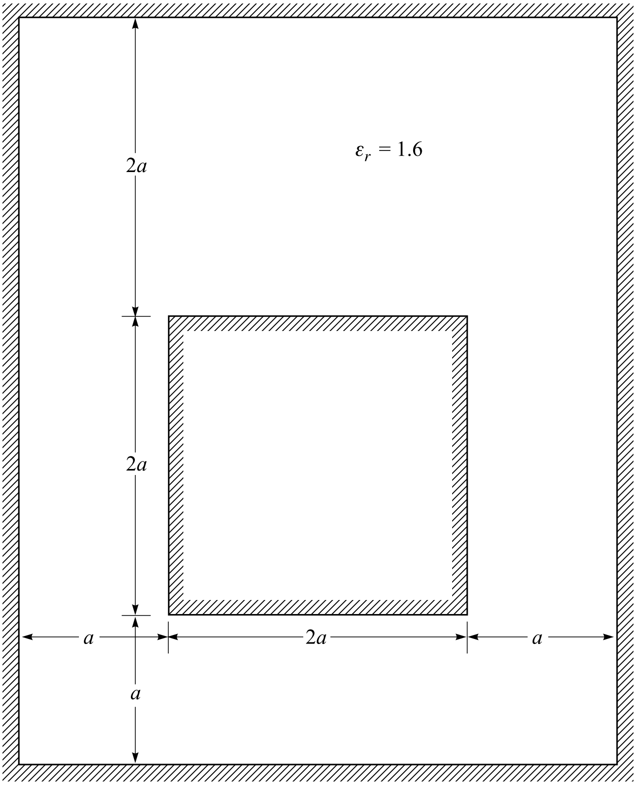

Figure 6.13 See Problem 6.21.

6.24 A potential field in free space is given in spherical coordinates as

$$V(r) = \begin{cases} [\rho_0/(6\epsilon_0)]\,[3a^2 - r^2] & (r \le a) \\ (a^3\rho_0)/(3\epsilon_0 r) & (r \ge a) \end{cases}$$

where ρ_0 and a are constants. (a) Use Poisson's equation to find the volume charge density everywhere. (b) Find the total charge present.

6.25 Let $V = 2xy^2z^3$ and $\epsilon = \epsilon_0$. Given point $P(1, 2, -1)$, find. (a) V at P; (b) **E** at P; (c) ρ_v at P; (d) the equation of the equipotential surface passing through P; (e) the equation of the streamline passing through P. (f) Does V satisfy Laplace's equation?

6.26 Given the spherically symmetric potential field in free space, $V = V_0 e^{-r/a}$, find. (a) ρ_v at $r = a$; (b) the electric field at $r = a$; (c) the total charge.

6.27 Let $V(x, y) = 4e^{2x} + f(x) - 3y^2$ in a region of free space where $\rho_v = 0$. It is known that both E_x and V are zero at the origin. Find $f(x)$ and $V(x, y)$.

6.28 Show that in a homogeneous medium of conductivity σ, the potential field V satisfies Laplace's equation if any volume charge density present does not vary with time.

6.29 Given the potential field $V = (A\rho^4 + B\rho^{-4})\sin 4\phi$: (*a*) Show that $\nabla^2 V = 0$. (*b*) Select A and B so that $V = 100$ V and $|\mathbf{E}| = 500$ V/m at $P(\rho = 1, \phi = 22.5^\circ, z = 2)$.

6.30 A parallel-plate capacitor has plates located at $z = 0$ and $z = d$. The region between plates is filled with a material that contains volume charge of uniform density ρ_0 C/m^3 and has permittivity ϵ. Both plates are held at ground potential. (*a*) Determine the potential field between plates. (*b*) Determine the electric field intensity $\mathbf{E}$ between plates. (*c*) Repeat parts (*a*) and (*b*) for the case of the plate at $z = d$ raised to potential V_0, with the $z = 0$ plate grounded.

6.31 Let $V = (\cos 2\phi)/\rho$ in free space. (*a*) Find the volume charge density at point $A(0.5, 60^\circ, 1)$. (*b*) Find the surface charge density on a conductor surface passing through the point $B(2, 30^\circ, 1)$.

6.32 A uniform volume charge has constant density $\rho_v = \rho_0$ C/m^3 and fills the region $r < a$, in which permittivity ϵ is assumed. A conducting spherical shell is located at $r = a$ and is held at ground potential. Find (*a*) the potential everywhere; (*b*) the electric field intensity, $\mathbf{E}$, everywhere.

6.33 The functions $V_1(\rho, \phi, z)$ and $V_2(\rho, \phi, z)$ both satisfy Laplace's equation in the region $a < \rho < b$, $0 \leq \phi < 2\pi$, $-L < z < L$; each is zero on the surfaces $\rho = b$ for $-L < z < L$; $z = -L$ for $a < \rho < b$; and $z = L$ for $a < \rho < b$; and each is 100 V on the surface $\rho = a$ for $-L < z < L$. (*a*) In the region specified, is Laplace's equation satisfied by the functions $V_1 + V_2$, $V_1 - V_2$, $V_1 + 3$, and $V_1 V_2$? (*b*) On the boundary surfaces specified, are the potential values given in this problem obtained from the functions $V_1 + V_2$, $V_1 - V_2$, $V_1 + 3$, and $V_1 V_2$? (*c*) Are the functions $V_1 + V_2$, $V_1 - V_2$, $V_1 + 3$, and $V_1 V_2$ identical with V_1?

6.34 Consider the parallel-plate capacitor of Problem 6.30, but this time the charged dielectric exists only between $z = 0$ and $z = b$, where $b < d$. Free space fills the region $b < z < d$. Both plates are at ground potential. By solving Laplace's *and* Poisson's equations, find (*a*) $V(z)$ for $0 < z < d$; (*b*) the electric field intensity for $0 < z < d$. No surface charge exists at $z = b$, so both V and $\mathbf{D}$ are continuous there.

6.35 The conducting planes $2x + 3y = 12$ and $2x + 3y = 18$ are at potentials of 100 V and 0, respectively. Let $\epsilon = \epsilon_0$ and find (*a*) V at $P(5, 2, 6)$; (*b*) $\mathbf{E}$ at P.

6.36 The derivation of Laplace's and Poisson's equations assumed constant permittivity, but there are cases of spatially varying permittivity in which the equations will still apply. Consider the vector identity, $\nabla \cdot (\psi \mathbf{G}) = \mathbf{G} \cdot \nabla \psi + \psi \nabla \cdot \mathbf{G}$, where ψ and $\mathbf{G}$ are scalar and vector functions, respectively.

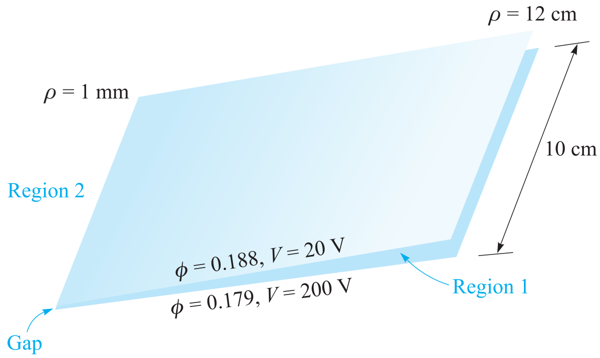

Figure 6.14 See Problem 6.39.

Determine a general rule on the allowed *directions* in which ϵ may vary with respect to the local electric field.

6.37 Coaxial conducting cylinders are located at $\rho = 0.5$ cm and $\rho = 1.2$ cm. The region between the cylinders is filled with a homogeneous perfect dielectric. If the inner cylinder is at 100 V and the outer at 0 V, find (*a*) the location of the 20 V equipotential surface; (*b*) $E_{\rho\,\max}$; (*c*) ϵ_r if the charge per meter length on the inner cylinder is 20 nC/m.

6.38 Repeat Problem 6.37, but with the dielectric only partially filling the volume, within $0 < \phi < \pi$, and with free space in the remaining volume.

6.39 The two conducting planes illustrated in Figure 6.14 are defined by $0.001 < \rho < 0.120$ m, $0 < z < 0.1$ m, $\phi = 0.179$ and 0.188 rad. The medium surrounding the planes is air. For Region 1, $0.179 < \phi < 0.188$; neglect fringing and find (*a*) $V(\phi)$; (*b*) $\mathbf{E}(\rho)$; (*c*) $\mathbf{D}(\rho)$; (*d*) ρ_s on the upper surface of the lower plane; (*e*) Q on the upper surface of the lower plane. (*f*) Repeat parts (*a*) through (*c*) for Region 2 by letting the location of the upper plane be $\phi = .188 - 2\pi$, and then find ρ_s and Q on the lower surface of the lower plane. (*g*) Find the total charge on the lower plane and the capacitance between the planes.

6.40 A parallel-plate capacitor is made using two circular plates of radius a, with the bottom plate on the xy plane, centered at the origin. The top plate is located at $z = d$, with its center on the z axis. Potential V_0 is on the top plate; the bottom plate is grounded. Dielectric having *radially dependent* permittivity fills the region between plates. The permittivity is given by $\epsilon(\rho) = \epsilon_0(1 + \rho^2/a^2)$. Find (*a*) $V(z)$; (*b*) $\mathbf{E}$; (*c*) Q; (*d*) C. This is a reprise of Problem 6.8, but it starts with Laplace's equation.

6.41 Concentric conducting spheres are located at $r = 5$ mm and $r = 20$ mm. The region between the spheres is filled with a perfect dielectric. If the inner sphere is at 100 V and the outer sphere is at 0 V (*a*) Find the location of the 20 V equipotential surface. (b) Find $E_{r,\max}$. (c) Find ϵ_r if the surface charge density on the inner sphere is 1.0 μC/m^2.

6.42 The hemisphere $0 < r < a$, $0 < \theta < \pi/2$, is composed of homogeneous conducting material of conductivity σ. The flat side of the hemisphere rests on a perfectly conducting plane. Now, the material within the conical region $0 < \theta < \alpha$, $0 < r < a$ is drilled out and replaced with material that is perfectly conducting. An air gap is maintained between the $r = 0$ tip of this new material and the plane. What resistance is measured between the two perfect conductors? Neglect fringing fields.

6.43 Two coaxial conducting cones have their vertices at the origin and the z axis as their axis. Cone A has the point $A(1, 0, 2)$ on its surface, while cone B has the point $B(0, 3, 2)$ on its surface. Let $V_A = 100$ V and $V_B = 20$ V. Find (a) α for each cone; (b) V at $P(1, 1, 1)$.

6.44 A potential field in free space is given as $V = 100 \ln \tan(\theta/2) + 50$ V. (a) Find the maximum value of $|\mathbf{E}_\theta|$ on the surface $\theta = 40^\circ$ for $0.1 < r < 0.8$ m, $60^\circ < \phi < 90^\circ$. ($b$) Describe the surface $V = 80$ V.

6.45 In free space, let $\rho_v = 200\epsilon_0/r^{2.4}$. ($a$) Use Poisson's equation to find $V(r)$ if it is assumed that $r^2 E_r \to 0$ when $r \to 0$, and also that $V \to 0$ as $r \to \infty$. (b) Now find $V(r)$ by using Gauss's law and a line integral.

6.46 By appropriate solution of Laplace's *and* Poisson's equations, determine the absolute potential at the center of a sphere of radius a, containing uniform volume charge of density ρ_0. Assume permittivity ϵ_0 everywhere. *Hint*: What must be true about the potential and the electric field at $r = 0$ and at $r = a$?

7

CHAPTER

The Steady Magnetic Field

At this point, the concept of a field should be a familiar one. Since we first accepted the experimental law of forces existing between two point charges and defined electric field intensity as the force per unit charge on a test charge in the presence of a second charge, we have discussed numerous fields. These fields possess no real physical basis, for physical measurements must always be in terms of the forces on the charges in the detection equipment. Those charges that are the source cause measurable forces to be exerted on other charges, which we may think of as detector charges. The fact that we attribute a field to the source charges and then determine the effect of this field on the detector charges amounts merely to a division of the basic problem into two parts for convenience.

We will begin our study of the magnetic field with a definition of the magnetic field itself and show how it arises from a current distribution. The effect of this field on other currents, or the second half of the physical problem, will be discussed in Chapter 8. As we did with the electric field, we confine our initial discussion to free-space conditions, and the effect of material media will also be saved for discussion in Chapter 8.

The relation of the steady magnetic field to its source is more complicated than is the relation of the electrostatic field to its source. We will find it necessary to accept several laws temporarily on faith alone. The proof of the laws does exist and is available on the Web site for the disbelievers or the more advanced student. ■

7.1 BIOT-SAVART LAW

The source of the steady magnetic field may be a permanent magnet, an electric field changing linearly with time, or a direct current. We will largely ignore the permanent magnet and save the time-varying electric field for a later discussion. Our present study will concern the magnetic field produced by a differential dc element in free space.

We may think of this differential current element as a vanishingly small section of a current-carrying filamentary conductor, where a filamentary conductor is the limiting

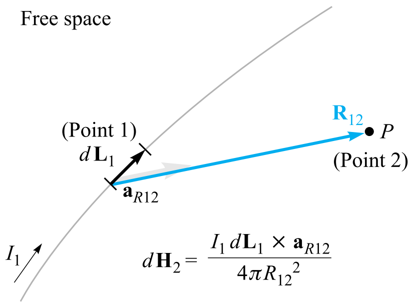

Figure 7.1 The law of Biot-Savart expresses the magnetic field intensity $d\mathbf{H}_2$ produced by a differential current element $I_1 d\mathbf{L}_1$. The direction of $d\mathbf{H}_2$ is into the page.

case of a cylindrical conductor of circular cross section as the radius approaches zero. We assume a current I flowing in a differential vector length of the filament $d\mathbf{L}$. The law of Biot-Savart[1] then states that at any point P the magnitude of the magnetic field intensity produced by the differential element is proportional to the product of the current, the magnitude of the differential length, and the sine of the angle lying between the filament and a line connecting the filament to the point P at which the field is desired; also, the magnitude of the magnetic field intensity is inversely proportional to the square of the distance from the differential element to the point P. The direction of the magnetic field intensity is normal to the plane containing the differential filament and the line drawn from the filament to the point P. Of the two possible normals, that one to be chosen is the one which is in the direction of progress of a right-handed screw turned from $d\mathbf{L}$ through the smaller angle to the line from the filament to P. Using rationalized mks units, the constant of proportionality is $1/4\pi$.

The *Biot-Savart law,* just described in some 150 words, may be written concisely using vector notation as

$$d\mathbf{H} = \frac{I\,d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = \frac{I\,d\mathbf{L} \times \mathbf{R}}{4\pi R^3} \tag{1}$$

The units of the *magnetic field intensity* $\mathbf{H}$ are evidently amperes per meter (A/m). The geometry is illustrated in Figure 7.1. Subscripts may be used to indicate the point to which each of the quantities in (1) refers. If we locate the current element at point 1 and describe the point P at which the field is to be determined as point 2, then

$$\boxed{d\mathbf{H}_2 = \frac{I_1 d\mathbf{L}_1 \times \mathbf{a}_{R12}}{4\pi R_{12}^2}} \tag{2}$$

[1] Biot and Savart were colleagues of Ampère, and all three were professors of physics at the Collège de France at one time or another. The Biot-Savart law was proposed in 1820.

The law of Biot-Savart is sometimes called *Ampère's law for the current element,* but we will retain the former name because of possible confusion with Ampère's circuital law, to be discussed later.

In some aspects, the Biot-Savart law is reminiscent of Coulomb's law when that law is written for a differential element of charge,

$$d\mathbf{E}_2 = \frac{dQ_1 \mathbf{a}_{R12}}{4\pi\epsilon_0 R_{12}^2}$$

Both show an inverse-square-law dependence on distance, and both show a linear relationship between source and field. The chief difference appears in the direction of the field.

It is impossible to check experimentally the law of Biot-Savart as expressed by (1) or (2) because the differential current element cannot be isolated. We have restricted our attention to direct currents only, so the charge density is not a function of time. The continuity equation in Section 5.2, Eq. (5),

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$$

therefore shows that

$$\nabla \cdot \mathbf{J} = 0$$

or upon applying the divergence theorem,

$$\oint_s \mathbf{J} \cdot d\mathbf{S} = 0$$

The total current crossing any closed surface is zero, and this condition may be satisfied only by assuming a current flow around a closed path. It is this current flowing in a closed circuit that must be our experimental source, not the differential element.

It follows that only the integral form of the Biot-Savart law can be verified experimentally,

$$\boxed{\mathbf{H} = \oint \frac{I\,d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}} \tag{3}$$

Equation (1) or (2), of course, leads directly to the integral form (3), but other differential expressions also yield the same integral formulation. Any term may be added to (1) whose integral around a closed path is zero. That is, any conservative field could be added to (1). The gradient of any scalar field always yields a conservative field, and we could therefore add a term ∇G to (1), where G is a general scalar field, without changing (3) in the slightest. This qualification on (1) or (2) is mentioned to show that if we later ask some foolish questions, not subject to any experimental check, concerning the force exerted by one *differential* current element on another, we should expect foolish answers.

The Biot-Savart law may also be expressed in terms of distributed sources, such as current density **J** and *surface current density* **K**. Surface current flows in a sheet of vanishingly small thickness, and the current density **J**, measured in amperes per square

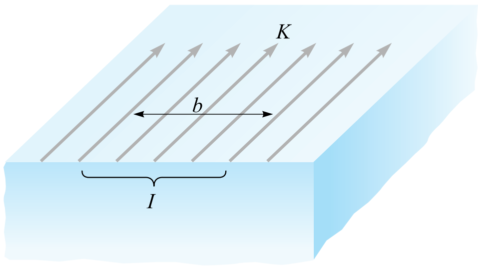

Figure 7.2 The total current I within a transverse width b, in which there is a *uniform* surface current density K, is Kb.

meter, is therefore infinite. Surface current density, however, is measured in amperes per meter width and designated by **K**. If the surface current density is uniform, the total current I in any width b is

$$I = Kb$$

where we assume that the width b is measured perpendicularly to the direction in which the current is flowing. The geometry is illustrated by Figure 7.2. For a nonuniform surface current density, integration is necessary:

$$\boxed{I = \int K dN} \tag{4}$$

where dN is a differential element of the path *across* which the current is flowing. Thus the differential current element $I\,d\mathbf{L}$, where $d\mathbf{L}$ is in the direction of the current, may be expressed in terms of surface current density **K** or current density **J**,

$$I\,d\mathbf{L} = \mathbf{K}\,dS = \mathbf{J}\,dv \tag{5}$$

and alternate forms of the Biot-Savart law obtained,

$$\boxed{\mathbf{H} = \int_s \frac{\mathbf{K} \times \mathbf{a}_R dS}{4\pi R^2}} \tag{6}$$

and

$$\boxed{\mathbf{H} = \int_{\text{vol}} \frac{\mathbf{J} \times \mathbf{a}_R dv}{4\pi R^2}} \tag{7}$$

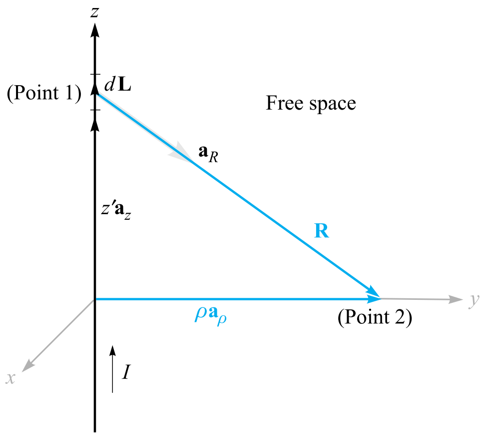

Figure 7.3 An infinitely long straight filament carrying a direct current I. The field at point 2 is $\mathbf{H} = (I/2\pi\rho)\mathbf{a}_\phi$.

We illustrate the application of the Biot-Savart law by considering an infinitely long straight filament. We apply (2) first and then integrate. This, of course, is the same as using the integral form (3) in the first place.[2]

Referring to Figure 7.3, we should recognize the symmetry of this field. No variation with z or with ϕ can exist. Point 2, at which we will determine the field, is therefore chosen in the $z = 0$ plane. The field point $\mathbf{r}$ is therefore $r = \rho\mathbf{a}_\rho$. The source point $\mathbf{r}'$ is given by $\mathbf{r}' = z'\mathbf{a}_z$, and therefore

$$\mathbf{R}_{12} = \mathbf{r} - \mathbf{r}' = \rho\mathbf{a}_\rho - z'\mathbf{a}_z$$

so that

$$\mathbf{a}_{R12} = \frac{\rho\mathbf{a}_\rho - z'\mathbf{a}_z}{\sqrt{\rho^2 + z'^2}}$$

We take $d\mathbf{L} = dz'\mathbf{a}_z$ and (2) becomes

$$d\mathbf{H}_2 = \frac{I\,dz'\mathbf{a}_z \times (\rho\mathbf{a}_\rho - z'\mathbf{a}_z)}{4\pi(\rho^2 + z'^2)^{3/2}}$$

Because the current is directed toward increasing values of z', the limits are $-\infty$ and ∞ on the integral, and we have

$$\mathbf{H}_2 = \int_{-\infty}^{\infty} \frac{I\,dz'\mathbf{a}_z \times (\rho\mathbf{a}_\rho - z'\mathbf{a}_z)}{4\pi(\rho^2 + z'^2)^{3/2}}$$

$$= \frac{I}{4\pi}\int_{-\infty}^{\infty} \frac{\rho dz'\mathbf{a}_\phi}{(\rho^2 + z'^2)^{3/2}}$$

[2] The closed path for the current may be considered to include a return filament parallel to the first filament and infinitely far removed. An outer coaxial conductor of infinite radius is another theoretical possibility. Practically, the problem is an impossible one, but we should realize that our answer will be quite accurate near a very long, straight wire having a distant return path for the current.

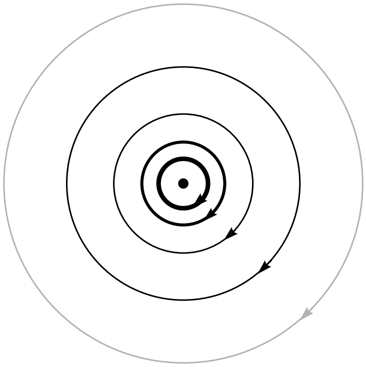

Figure 7.4 The streamlines of the magnetic field intensity about an infinitely long straight filament carrying a direct current I. The direction of I is into the page.

At this point the unit vector $\mathbf{a}_\phi$ under the integral sign should be investigated, for it is not always a constant, as are the unit vectors of the rectangular coordinate system. A vector is constant when its magnitude and direction are both constant. The unit vector certainly has constant magnitude, but its direction may change. Here $\mathbf{a}_\phi$ changes with the coordinate ϕ but not with ρ or z. Fortunately, the integration here is with respect to z', and $\mathbf{a}_\phi$ is a constant and may be removed from under the integral sign,

$$\mathbf{H}_2 = \frac{I\rho\mathbf{a}_\phi}{4\pi}\int_{-\infty}^{\infty}\frac{dz'}{(\rho^2 + z'^2)^{3/2}}$$
$$= \frac{I\rho\mathbf{a}_\phi}{4\pi}\,\frac{z'}{\rho^2\sqrt{\rho^2 + z'^2}}\Bigg|_{-\infty}^{\infty}$$

and

$$\boxed{\mathbf{H}_2 = \frac{I}{2\pi\rho}\mathbf{a}_\phi} \tag{8}$$

The magnitude of the field is not a function of ϕ or z, and it varies inversely with the distance from the filament. The direction of the magnetic-field-intensity vector is circumferential. The streamlines are therefore circles about the filament, and the field may be mapped in cross section as in Figure 7.4.

The separation of the streamlines is proportional to the radius, or inversely proportional to the magnitude of $\mathbf{H}$. To be specific, the streamlines have been drawn with curvilinear squares in mind. As yet, we have no name for the family of lines[3] that are perpendicular to these circular streamlines, but the spacing of the streamlines has

[3] If you can't wait, see Section 7.6.

been adjusted so that the addition of this second set of lines will produce an array of curvilinear squares.

A comparison of Figure 7.4 with the map of the *electric* field about an infinite line *charge* shows that the streamlines of the magnetic field correspond exactly to the equipotentials of the electric field, and the unnamed (and undrawn) perpendicular family of lines in the magnetic field corresponds to the streamlines of the electric field. This correspondence is not an accident, but there are several other concepts which must be mastered before the analogy between electric and magnetic fields can be explored more thoroughly.

Using the Biot-Savart law to find **H** is in many respects similar to the use of Coulomb's law to find **E**. Each requires the determination of a moderately complicated integrand containing vector quantities, followed by an integration. When we were concerned with Coulomb's law we solved a number of examples, including the fields of the point charge, line charge, and sheet of charge. The law of Biot-Savart can be used to solve analogous problems in magnetic fields, and some of these problems appear as exercises at the end of the chapter rather than as examples here.

One useful result is the field of the finite-length current element, shown in Figure 7.5. It turns out (see Problem 7.8 at the end of the chapter) that **H** is most easily expressed in terms of the angles α_1 and α_2, as identified in the figure. The result is

$$\mathbf{H} = \frac{I}{4\pi\rho}(\sin\alpha_2 - \sin\alpha_1)\mathbf{a}_\phi \tag{9}$$

If one or both ends are below point 2, then α_1 is or both α_1 and α_2 are negative.

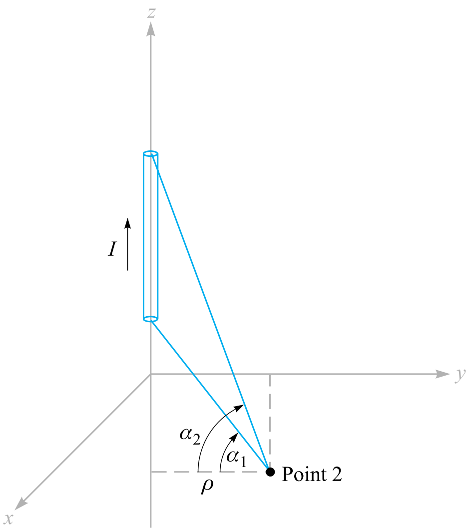

Figure 7.5 The magnetic field intensity caused by a finite-length current filament on the z axis is $(I/4\pi\rho)(\sin\alpha_2 - \sin\alpha_1)\mathbf{a}_\phi$.

Equation (9) may be used to find the magnetic field intensity caused by current filaments arranged as a sequence of straight-line segments.

EXAMPLE 7.1

As a numerical example illustrating the use of (9), we determine $\mathbf{H}$ at $P_2(0.4, 0.3, 0)$ in the field of an 8. A filamentary current is directed inward from infinity to the origin on the positive x axis, and then outward to infinity along the y axis. This arrangement is shown in Figure 7.6.

Solution. We first consider the semi-infinite current on the x axis, identifying the two angles, $\alpha_{1x} = -90^\circ$ and $\alpha_{2x} = \tan^{-1}(0.4/0.3) = 53.1^\circ$. The radial distance ρ is measured from the x axis, and we have $\rho_x = 0.3$. Thus, this contribution to $\mathbf{H}_2$ is

$$\mathbf{H}_{2(x)} = \frac{8}{4\pi(0.3)}(\sin 53.1^\circ + 1)\mathbf{a}_\phi = \frac{2}{0.3\pi}(1.8)\mathbf{a}_\phi = \frac{12}{\pi}\mathbf{a}_\phi$$

The unit vector $\mathbf{a}_\phi$ must also be referred to the x axis. We see that it becomes $-\mathbf{a}_z$. Therefore,

$$\mathbf{H}_{2(x)} = -\frac{12}{\pi}\mathbf{a}_z \text{ A/m}$$

For the current on the y axis, we have $\alpha_{1y} = -\tan^{-1}(0.3/0.4) = -36.9^\circ$, $\alpha_{2y} = 90^\circ$, and $\rho_y = 0.4$. It follows that

$$\mathbf{H}_{2(y)} = \frac{8}{4\pi(0.4)}(1 + \sin 36.9^\circ)(-\mathbf{a}_z) = -\frac{8}{\pi}\mathbf{a}_z \text{ A/m}$$

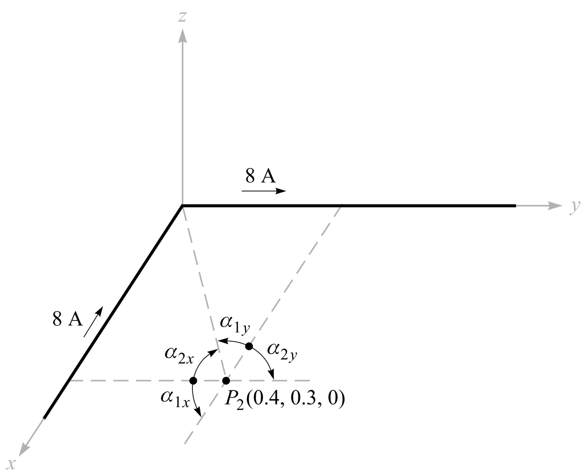

Figure 7.6 The individual fields of two semi-infinite current segments are found by (9) and added to obtain H_2 at P_2.

Adding these results, we have

$$\mathbf{H}_2 = \mathbf{H}_{2(x)} + \mathbf{H}_{2(y)} = -\frac{20}{\pi}\mathbf{a}_z = -6.37\mathbf{a}_z \text{ A/m}$$

D7.1. Given the following values for P_1, P_2, and $I_1 \Delta L_1$, calculate $\Delta\mathbf{H}_2$: (*a*) $P_1(0, 0, 2)$, $P_2(4, 2, 0)$, $2\pi\mathbf{a}_z\,\mu\text{A}\cdot\text{m}$; (*b*) $P_1(0, 2, 0)$, $P_2(4, 2, 3)$, $2\pi\mathbf{a}_z\,\mu\text{A}\cdot\text{m}$; (*c*) $P_1(1, 2, 3)$, $P_2(-3, -1, 2)$, $2\pi(-\mathbf{a}_x + \mathbf{a}_y + 2\mathbf{a}_z)\,\mu\text{A}\cdot\text{m}$.

Ans. $-8.51\mathbf{a}_x + 17.01\mathbf{a}_y$ nA/m; $16\mathbf{a}_y$ nA/m; $18.9\mathbf{a}_x - 33.9\mathbf{a}_y + 26.4\mathbf{a}_z$ nA/m

D7.2. A current filament carrying 15 A in the $\mathbf{a}_z$ direction lies along the entire z axis. Find $\mathbf{H}$ in rectangular coordinates at: (*a*) $P_A(\sqrt{20}, 0, 4)$; (*b*) $P_B(2, -4, 4)$.

Ans. $0.534\mathbf{a}_y$ A/m; $0.477\mathbf{a}_x + 0.239\mathbf{a}_y$ A/m

7.2 AMPÈRE'S CIRCUITAL LAW

After solving a number of simple electrostatic problems with Coulomb's law, we found that the same problems could be solved much more easily by using Gauss's law whenever a high degree of symmetry was present. Again, an analogous procedure exists in magnetic fields. Here, the law that helps us solve problems more easily is known as *Ampère's circuital*[4] *law,* sometimes called Ampère's work law. This law may be derived from the Biot-Savart law (see Section 7.7).

Ampère's circuital law states that the line integral of **H** about any *closed* path is exactly equal to the direct current enclosed by that path,

$$\boxed{\oint \mathbf{H} \cdot d\mathbf{L} = I} \tag{10}$$

We define positive current as flowing in the direction of advance of a right-handed screw turned in the direction in which the closed path is traversed.

Referring to Figure 7.7, which shows a circular wire carrying a direct current I, the line integral of **H** about the closed paths lettered a and b results in an answer of I; the integral about the closed path c which passes through the conductor gives an answer less than I and is exactly that portion of the total current that is enclosed by the path c. Although paths a and b give the same answer, the integrands are, of course, different. The line integral directs us to multiply the component of **H** in the direction of the path by a small increment of path length at one point of the path, move along the path to the next incremental length, and repeat the process, continuing until the path is completely traversed. Because **H** will generally vary from point to point, and because paths a and b are not alike, the contributions to the integral made by, say,

[4] The preferred pronunciation puts the accent on "circ-."

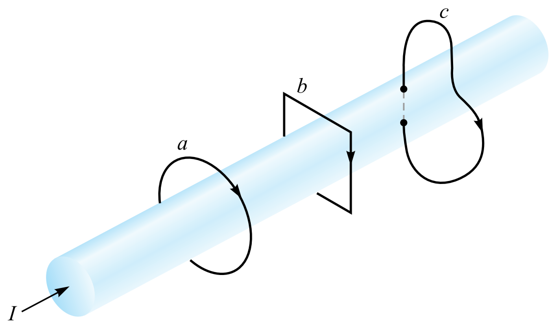

Figure 7.7 A conductor has a total current *I*. The line integral of **H** about the closed paths *a* and *b* is equal to *I*, and the integral around path *c* is less than *I*, since the entire current is not enclosed by the path.

each micrometer of path length are quite different. Only the final answers are the same.

We should also consider exactly what is meant by the expression "current enclosed by the path." Suppose we solder a circuit together after passing the conductor once through a rubber band, which we use to represent the closed path. Some strange and formidable paths can be constructed by twisting and knotting the rubber band, but if neither the rubber band nor the conducting circuit is broken, the current enclosed by the path is that carried by the conductor. Now replace the rubber band by a circular ring of spring steel across which is stretched a rubber sheet. The steel loop forms the closed path, and the current-carrying conductor must pierce the rubber sheet if the current is to be enclosed by the path. Again, we may twist the steel loop, and we may also deform the rubber sheet by pushing our fist into it or folding it in any way we wish. A single current-carrying conductor still pierces the sheet once, and this is the true measure of the current enclosed by the path. If we should thread the conductor once through the sheet from front to back and once from back to front, the total current enclosed by the path is the algebraic sum, which is zero.

In more general language, given a closed path, we recognize this path as the perimeter of an infinite number of surfaces (not closed surfaces). Any current-carrying conductor enclosed by the path must pass through every one of these surfaces once. Certainly some of the surfaces may be chosen in such a way that the conductor pierces them twice in one direction and once in the other direction, but the algebraic total current is still the same.

We will find that the nature of the closed path is usually extremely simple and can be drawn on a plane. The simplest surface is, then, that portion of the plane enclosed by the path. We need merely find the total current passing through this region of the plane.

The application of Gauss's law involves finding the total charge enclosed by a closed surface; the application of Ampère's circuital law involves finding the total current enclosed by a closed path.

Let us again find the magnetic field intensity produced by an infinitely long filament carrying a current I. The filament lies on the z axis in free space (as in Figure 7.3), and the current flows in the direction given by $\mathbf{a}_z$. Symmetry inspection comes first, showing that there is no variation with z or ϕ. Next we determine which components of $\mathbf{H}$ are present by using the Biot-Savart law. Without specifically using the cross product, we may say that the direction of $d\mathbf{H}$ is perpendicular to the plane conaining $d\mathbf{L}$ and $\mathbf{R}$ and therefore is in the direction of $\mathbf{a}_\phi$. Hence the only component of $\mathbf{H}$ is H_ϕ, and it is a function only of ρ.

We therefore choose a path, to any section of which $\mathbf{H}$ is either perpendicular or tangential, and along which H is constant. The first requirement (perpendicularity or tangency) allows us to replace the dot product of Ampère's circuital law with the product of the scalar magnitudes, except along that portion of the path where $\mathbf{H}$ is normal to the path and the dot product is zero; the second requirement (constancy) then permits us to remove the magnetic field intensity from the integral sign. The integration required is usually trivial and consists of finding the length of that portion of the path to which $\mathbf{H}$ is parallel.

In our example, the path must be a circle of radius ρ, and Ampère's circuital law becomes

$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_0^{2\pi} H_\phi \rho d\phi = H_\phi \rho \int_0^{2\pi} d\phi = H_\phi 2\pi\rho = I$$

or

$$H_\phi = \frac{I}{2\pi\rho}$$

as before.

As a second example of the application of Ampère's circuital law, consider an infinitely long coaxial transmission line carrying a uniformly distributed total current I in the center conductor and $-I$ in the outer conductor. The line is shown in Figure 7.8*a*. Symmetry shows that H is not a function of ϕ or z. In order to determine the components present, we may use the results of the previous example by considering the solid conductors as being composed of a large number of filaments. No filament has a z component of $\mathbf{H}$. Furthermore, the H_ρ component at $\phi = 0^\circ$, produced by one filament located at $\rho = \rho_1$, $\phi = \phi_1$, is canceled by the H_ρ component produced by a symmetrically located filament at $\rho = \rho_1$, $\phi = -\phi_1$. This symmetry is illustrated by Figure 7.8*b*. Again we find only an H_ϕ component which varies with ρ.

A circular path of radius ρ, where ρ is larger than the radius of the inner conductor but less than the inner radius of the outer conductor, then leads immediately to

$$H_\phi = \frac{I}{2\pi\rho} \quad (a < \rho < b)$$

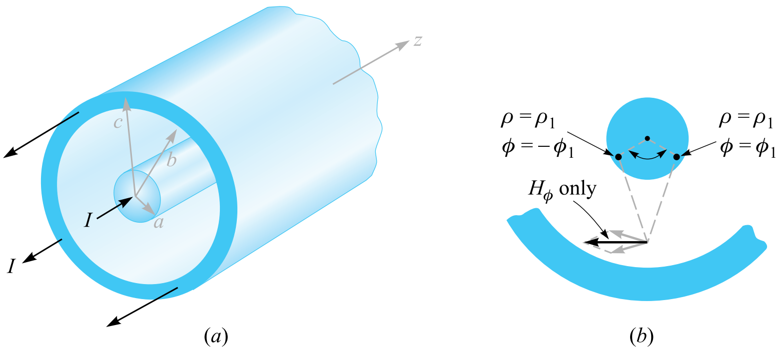

Figure 7.8 (*a*) Cross section of a coaxial cable carrying a uniformly distributed current I in the inner conductor and $-I$ in the outer conductor. The magnetic field at any point is most easily determined by applying Ampère's circuital law about a circular path. (*b*) Current filaments at $\rho = \rho_1, \phi = \pm\phi_1$, produces H_ρ components which cancel. For the total field, $\mathbf{H} = H_\phi \mathbf{a}_\phi$.

Illustrations

If we choose ρ smaller than the radius of the inner conductor, the current enclosed is

$$I_{\text{encl}} = I\frac{\rho^2}{a^2}$$

and

$$2\pi\rho H_\phi = I\frac{\rho^2}{a^2}$$

or

$$H_\phi = \frac{I\rho}{2\pi a^2} \quad (\rho < a)$$

If the radius ρ is larger than the outer radius of the outer conductor, no current is enclosed and

$$H_\phi = 0 \quad (\rho > c)$$

Finally, if the path lies within the outer conductor, we have

$$2\pi\rho H_\phi = I - I\left(\frac{\rho^2 - b^2}{c^2 - b^2}\right)$$

$$H_\phi = \frac{I}{2\pi\rho}\frac{c^2 - \rho^2}{c^2 - b^2} \quad (b < \rho < c)$$

The magnetic-field-strength variation with radius is shown in Figure 7.9 for a coaxial cable in which $b = 3a$, $c = 4a$. It should be noted that the magnetic field intensity **H** is continuous at all the conductor boundaries. In other words, a slight increase in the radius of the closed path does not result in the enclosure of a tremendously different current. The value of H_ϕ shows no sudden jumps.

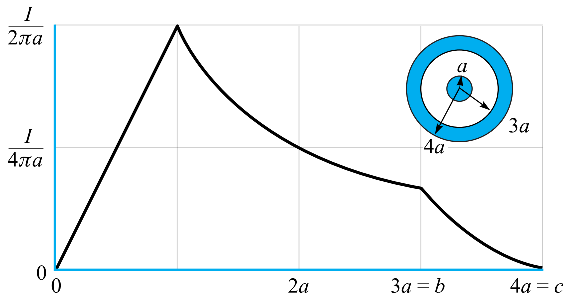

Figure 7.9 The magnetic field intensity as a function of radius in an infinitely long coaxial transmission line with the dimensions shown.

The external field is zero. This, we see, results from equal positive and negative currents enclosed by the path. Each produces an external field of magnitude $I/2\pi\rho$, but complete cancellation occurs. This is another example of "shielding"; such a coaxial cable carrying large currents would, in principle, not produce any noticeable effect in an adjacent circuit.

As a final example, let us consider a sheet of current flowing in the positive y direction and located in the $z = 0$ plane. We may think of the return current as equally divided between two distant sheets on either side of the sheet we are considering. A sheet of uniform surface current density $\mathbf{K} = K_y\mathbf{a}_y$ is shown in Figure 7.10. $\mathbf{H}$ cannot vary with x or y. If the sheet is subdivided into a number of filaments, it is evident that no filament can produce an H_y component. Moreover, the Biot-Savart law shows that the contributions to H_z produced by a symmetrically located pair of filaments cancel. Thus, H_z is zero also; only an H_x component is present. We therefore choose the path 1-1′-2′-2-1 composed of straight-line segments that are either parallel or

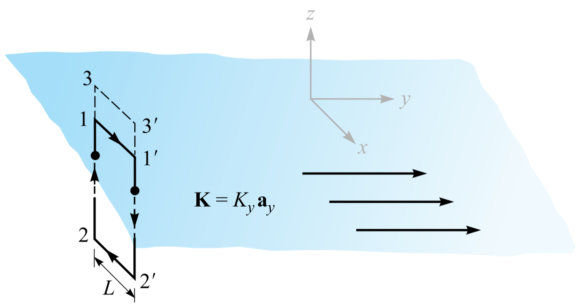

Figure 7.10 A uniform sheet of surface current $\mathbf{K} = K_y\mathbf{a}_y$ in the $z = 0$ plane. $\mathbf{H}$ may be found by applying Ampère's circuital law about the paths 1-1′-2′-2-1 and 3-3′-2′-2-3.

perpendicular to H_x. Ampère's circuital law gives

$$H_{x1}L + H_{x2}(-L) = K_y L$$

or

$$H_{x1} - H_{x2} = K_y$$

If the path 3-3′-2′-2-3 is now chosen, the same current is enclosed, and

$$H_{x3} - H_{x2} = K_y$$

and therefore

$$H_{x3} = H_{x1}$$

It follows that H_x is the same for all positive z. Similarly, H_x is the same for all negative z. Because of the symmetry, then, the magnetic field intensity on one side of the current sheet is the negative of that on the other. Above the sheet,

$$H_x = \tfrac{1}{2}K_y \quad (z > 0)$$

while below it

$$H_x = -\tfrac{1}{2}K_y \quad (z < 0)$$

Letting $\mathbf{a}_N$ be a unit vector normal (outward) to the current sheet, the result may be written in a form correct for all z as

$$\boxed{\mathbf{H} = \tfrac{1}{2}\mathbf{K} \times \mathbf{a}_N} \tag{11}$$

If a second sheet of current flowing in the opposite direction, $\mathbf{K} = -K_y\mathbf{a}_y$, is placed at $z = h$, (11) shows that the field in the region between the current sheets is

$$\boxed{\mathbf{H} = \mathbf{K} \times \mathbf{a}_N \quad (0 < z < h)} \tag{12}$$

and is zero elsewhere,

$$\boxed{\mathbf{H} = 0 \quad (z < 0, z > h)} \tag{13}$$

The most difficult part of the application of Ampère's circuital law is the determination of the components of the field that are present. The surest method is the logical application of the Biot-Savart law and a knowledge of the magnetic fields of simple form.

Problem 7.13 at the end of this chapter outlines the steps involved in applying Ampère's circuital law to an infinitely long solenoid of radius a and uniform current density $K_a\mathbf{a}_\phi$, as shown in Figure 7.11a. For reference, the result is

$$\mathbf{H} = K_a\mathbf{a}_z \quad (\rho < a) \tag{14a}$$

$$\mathbf{H} = 0 \quad (\rho > a) \tag{14b}$$

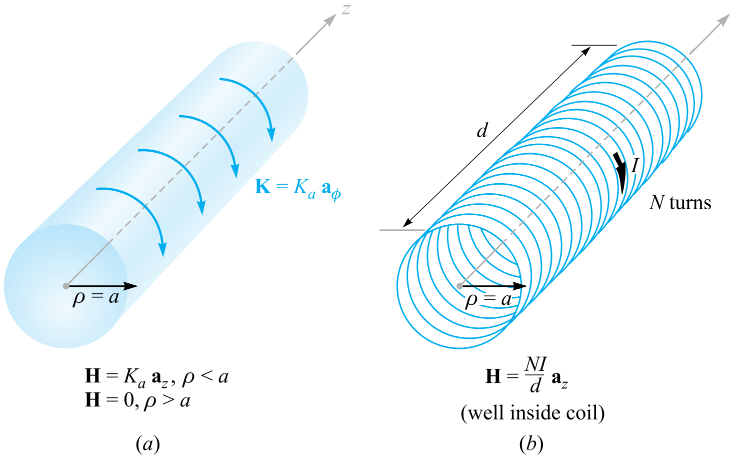

Figure 7.11 (*a*) An ideal solenoid of infinite length with a circular current sheet $\mathbf{K} = K_a\mathbf{a}_\phi$. (*b*) An N-turn solenoid of finite length d.

If the solenoid has a finite length d and consists of N closely wound turns of a filament that carries a current I (Figure 7.11*b*), then the field at points well within the solenoid is given closely by

$$\mathbf{H} = \frac{NI}{d}\mathbf{a}_z \quad \text{(well within the solenoid)} \tag{15}$$

The approximation is useful it if is not applied closer than two radii to the open ends, nor closer to the solenoid surface than twice the separation between turns.

For the toroids shown in Figure 7.12, it can be shown that the magnetic field intensity for the ideal case, Figure 7.12*a*, is

$$\mathbf{H} = K_a\frac{\rho_0 - a}{\rho}\mathbf{a}_\phi \quad \text{(inside toroid)} \tag{16a}$$

$$\mathbf{H} = 0 \quad \text{(outside)} \tag{16b}$$

For the N-turn toroid of Figure 7.12*b*, we have the good approximations,

$$\mathbf{H} = \frac{NI}{2\pi\rho}\mathbf{a}_\phi \quad \text{(inside toroid)} \tag{17a}$$

$$\mathbf{H} = 0 \quad \text{(outside)} \tag{17b}$$

as long as we consider points removed from the toroidal surface by several times the separation between turns.

Toroids having rectangular cross sections are also treated quite readily, as you can see for yourself by trying Problem 7.14.

Accurate formulas for solenoids, toroids, and coils of other shapes are available in Section 2 of the *Standard Handbook for Electrical Engineers* (see References for Chapter 5).

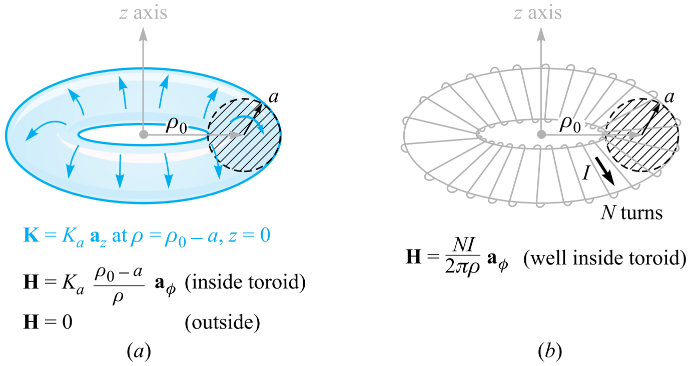

Figure 7.12 (*a*) An ideal toroid carrying a surface current **K** in the direction shown. (*b*) An *N*-turn toroid carrying a filamentary current *I*.

D7.3. Express the value of **H** in rectangular components at $P(0, 0.2, 0)$ in the field of: (*a*) a current filament, 2.5 A in the $\mathbf{a}_z$ direction at $x = 0.1$, $y = 0.3$; (*b*) a coax, centered on the z axis, with $a = 0.3$, $b = 0.5$, $c = 0.6$, $I = 2.5$ A in the $\mathbf{a}_z$ direction in the center conductor; (*c*) three current sheets, $2.7\mathbf{a}_x$ A/m at $y = 0.1$, $-1.4\mathbf{a}_x$ A/m at $y = 0.15$, and $-1.3\mathbf{a}_x$ A/m at $y = 0.25$.

Ans. $1.989\mathbf{a}_x - 1.989\mathbf{a}_y$ A/m; $-0.884\mathbf{a}_x$ A/m; $1.300\mathbf{a}_z$ A/m

7.3 CURL

We completed our study of Gauss's law by applying it to a differential volume element and were led to the concept of divergence. We now apply Ampère's circuital law to the perimeter of a differential surface element and discuss the third and last of the special derivatives of vector analysis, the curl. Our objective is to obtain the point form of Ampère's circuital law.

Again we choose rectangular coordinates, and an incremental closed path of sides Δx and Δy is selected (Figure 7.13). We assume that some current, as yet unspecified, produces a reference value for **H** at the *center* of this small rectangle,

$$\mathbf{H}_0 = H_{x0}\mathbf{a}_x + H_{y0}\mathbf{a}_y + H_{z0}\mathbf{a}_z$$

The closed line integral of **H** about this path is then approximately the sum of the four values of $\mathbf{H}\cdot\Delta\mathbf{L}$ on each side. We choose the direction of traverse as 1-2-3-4-1, which corresponds to a current in the $\mathbf{a}_z$ direction, and the first contribution is therefore

$$(\mathbf{H}\cdot\Delta\mathbf{L})_{1-2} = H_{y,1-2}\Delta y$$

The value of H_y *on this section* of the path may be given in terms of the reference value H_{y0} at the center of the rectangle, the rate of change of H_y with x, and the

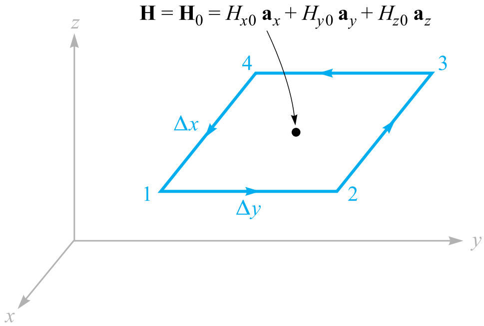

Figure 7.13 An incremental closed path in rectangular coordinates is selected for the application of Ampère's circuital law to determine the spatial rate of change of **H**.

distance $\Delta x/2$ from the center to the midpoint of side 1–2:

$$H_{y,1-2} \doteq H_{y0} + \frac{\partial H_y}{\partial x}\left(\frac{1}{2}\Delta x\right)$$

Thus

$$(\mathbf{H}\cdot\Delta\mathbf{L})_{1-2} \doteq \left(H_{y0} + \frac{1}{2}\frac{\partial H_y}{\partial x}\Delta x\right)\Delta y$$

Along the next section of the path we have

$$(\mathbf{H}\cdot\Delta\mathbf{L})_{2-3} \doteq H_{x,2-3}(-\Delta x) \doteq -\left(H_{x0} + \frac{1}{2}\frac{\partial H_x}{\partial y}\Delta y\right)\Delta x$$

Continuing for the remaining two segments and adding the results,

$$\oint \mathbf{H}\cdot d\mathbf{L} \doteq \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\Delta x\,\Delta y$$

By Ampère's circuital law, this result must be equal to the current enclosed by the path, or the current crossing any surface bounded by the path. If we assume a general current density **J**, the enclosed current is then $\Delta I \doteq J_z \Delta x \Delta y$, and

$$\oint \mathbf{H}\cdot d\mathbf{L} \doteq \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\Delta x\,\Delta y \doteq J_z\,\Delta x\,\Delta y$$

or

$$\frac{\oint \mathbf{H}\cdot d\mathbf{L}}{\Delta x\,\Delta y} \doteq \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \doteq J_z$$

As we cause the closed path to shrink, the preceding expression becomes more nearly exact, and in the limit we have the equality

$$\lim_{\Delta x,\Delta y\to 0} \frac{\oint \mathbf{H}\cdot d\mathbf{L}}{\Delta x\,\Delta y} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z \tag{18}$$

After beginning with Ampère's circuital law equating the closed line integral of **H** to the current enclosed, we have now arrived at a relationship involving the closed line integral of **H** *per unit area* enclosed and the current *per unit area* enclosed, or current density. We performed a similar analysis in passing from the integral form of Gauss's law, involving flux through a closed surface and charge enclosed, to the point form, relating flux through a closed surface *per unit volume* enclosed and charge *per unit volume* enclosed, or volume charge density. In each case a limit is necessary to produce an equality.

If we choose closed paths that are oriented perpendicularly to each of the remaining two coordinate axes, analogous processes lead to expressions for the x and y components of the current density,

$$\lim_{\Delta y, \Delta z \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta y \Delta z} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x \tag{19}$$

and

$$\lim_{\Delta z, \Delta x \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta z \Delta x} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y \tag{20}$$

Comparing (18)–(20), we see that a component of the current density is given by the limit of the quotient of the closed line integral of **H** about a small path in a plane normal to that component and of the area enclosed as the path shrinks to zero. This limit has its counterpart in other fields of science and long ago received the name of *curl*. The curl of any vector is a vector, and any component of the curl is given by the limit of the quotient of the closed line integral of the vector about a small path in a plane normal to that component desired and the area enclosed, as the path shrinks to zero. It should be noted that this definition of curl does not refer specifically to a particular coordinate system. The mathematical form of the definition is

$$\boxed{(\text{curl } \mathbf{H})_N = \lim_{\Delta S_N \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta S_N}} \tag{21}$$

where ΔS_N is the planar area enclosed by the closed line integral. The N subscript indicates that the component of the curl is that component which is *normal* to the surface enclosed by the closed path. It may represent any component in any coordinate system.

In rectangular coordinates, the definition (21) shows that the x, y, and z components of the curl **H** are given by (18)–(20), and therefore

$$\boxed{\text{curl } \mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) \mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) \mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \mathbf{a}_z} \tag{22}$$

This result may be written in the form of a determinant,

$$\text{curl}\,\mathbf{H} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} \tag{23}$$

and may also be written in terms of the vector operator,

$$\text{curl}\,\mathbf{H} = \nabla \times \mathbf{H} \tag{24}$$

Equation (22) is the result of applying the definition (21) to the rectangular coordinate system. We obtained the z component of this expression by evaluating Ampère's circuital law about an incremental path of sides Δx and Δy, and we could have obtained the other two components just as easily by choosing the appropriate paths. Equation (23) is a neat method of storing the rectangular coordinate expression for curl; the form is symmetrical and easily remembered. Equation (24) is even more concise and leads to (22) upon applying the definitions of the cross product and vector operator.

The expressions for curl $\mathbf{H}$ in cylindrical and spherical coordinates are derived in Appendix A by applying the definition (21). Although they may be written in determinant form, as explained there, the determinants do not have one row of unit vectors on top and one row of components on the bottom, and they are not easily memorized. For this reason, the curl expansions in cylindrical and spherical coordinates that follow here and appear inside the back cover are usually referred to whenever necessary.

$$\begin{aligned} \nabla \times \mathbf{H} = & \left(\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right)\mathbf{a}_\rho + \left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho}\right)\mathbf{a}_\phi \\ & + \left(\frac{1}{\rho}\frac{\partial(\rho H_\phi)}{\partial \rho} - \frac{1}{\rho}\frac{\partial H_\rho}{\partial \phi}\right)\mathbf{a}_z \quad \text{(cylindrical)} \end{aligned} \tag{25}$$

$$\begin{aligned} \nabla \times \mathbf{H} = & \frac{1}{r\sin\theta}\left(\frac{\partial(H_\phi \sin\theta)}{\partial \theta} - \frac{\partial H_\theta}{\partial \phi}\right)\mathbf{a}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial H_r}{\partial \phi} - \frac{\partial(rH_\phi)}{\partial r}\right)\mathbf{a}_\theta \\ & + \frac{1}{r}\left(\frac{\partial(rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta}\right)\mathbf{a}_\phi \quad \text{(spherical)} \end{aligned} \tag{26}$$

Although we have described curl as a line integral per unit area, this does not provide everyone with a satisfactory physical picture of the nature of the curl operation, for the closed line integral itself requires physical interpretation. This integral was first met in the electrostatic field, where we saw that $\oint \mathbf{E} \cdot d\mathbf{L} = 0$. Inasmuch as the integral was zero, we did not belabor the physical picture. More recently we have discussed the closed line integral of $\mathbf{H}$, $\oint \mathbf{H} \cdot d\mathbf{L} = I$. Either of these closed line integrals is also known by the name of *circulation,* a term borrowed from the field of fluid dynamics.

Figure 7.14 (*a*) The curl meter shows a component of the curl of the water velocity into the page. (*b*) The curl of the magnetic field intensity about an infinitely long filament is shown.

The circulation of **H**, or $\oint \mathbf{H} \cdot d\mathbf{L}$, is obtained by multiplying the component of **H** parallel to the specified closed path at each point along it by the differential path length and summing the results as the differential lengths approach zero and as their number becomes infinite. We do not require a vanishingly small path. Ampère's circuital law tells us that if **H** does possess circulation about a given path, then current passes through this path. In electrostatics we see that the circulation of **E** is zero about every path, a direct consequence of the fact that zero work is required to carry a charge around a closed path.

We may describe curl as *circulation per unit area*. The closed path is vanishingly small, and curl is defined at a point. The curl of **E** must be zero, for the circulation is zero. The curl of **H** is not zero, however; the circulation of **H** per unit area is the current density by Ampère's circuital law [or (18), (19), and (20)].

Skilling[5] suggests the use of a very small paddle wheel as a "curl meter." Our vector quantity, then, must be thought of as capable of applying a force to each blade of the paddle wheel, the force being proportional to the component of the field normal to the surface of that blade. To test a field for curl, we dip our paddle wheel into the field, with the axis of the paddle wheel lined up with the direction of the component of curl desired, and note the action of the field on the paddle. No rotation means no curl; larger angular velocities mean greater values of the curl; a reversal in the direction of spin means a reversal in the sign of the curl. To find the direction of the vector curl and not merely to establish the presence of any particular component, we should place our paddle wheel in the field and hunt around for the orientation which produces the greatest torque. The direction of the curl is then along the axis of the paddle wheel, as given by the right-hand rule.

As an example, consider the flow of water in a river. Figure 7.14*a* shows the longitudinal section of a wide river taken at the middle of the river. The water velocity is zero at the bottom and increases linearly as the surface is approached. A paddle wheel placed in the position shown, with its axis perpendicular to the paper, will turn in a clockwise direction, showing the presence of a component of curl in the direction

[5] See the References at the end of the chapter.

of an inward normal to the surface of the page. If the velocity of water does not change as we go up- or downstream and also shows no variation as we go across the river (or even if it decreases in the same fashion toward either bank), then this component is the only component present at the center of the stream, and the curl of the water velocity has a direction into the page.

In Figure 7.14*b*, the streamlines of the magnetic field intensity about an infinitely long filamentary conductor are shown. The curl meter placed in this field of curved lines shows that a larger number of blades have a clockwise force exerted on them but that this force is in general smaller than the counterclockwise force exerted on the smaller number of blades closer to the wire. It seems possible that if the curvature of the streamlines is correct and also if the variation of the field strength is just right, the net torque on the paddle wheel may be zero. Actually, the paddle wheel does not rotate in this case, for since $\mathbf{H} = (I/2\pi\rho)\mathbf{a}_\phi$, we may substitute into (25) obtaining

$$\text{curl } \mathbf{H} = -\frac{\partial H_\phi}{\partial z}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial(\rho H_\phi)}{\partial \rho}\mathbf{a}_z = 0$$

EXAMPLE 7.2

As an example of the evaluation of curl $\mathbf{H}$ from the definition and of the evaluation of another line integral, suppose that $\mathbf{H} = 0.2z^2\mathbf{a}_x$ for $z > 0$, and $\mathbf{H} = 0$ elsewhere, as shown in Figure 7.15. Calculate $\oint \mathbf{H} \cdot d\mathbf{L}$ about a square path with side d, centered at $(0, 0, z_1)$ in the $y = 0$ plane where $z_1 > d/2$.

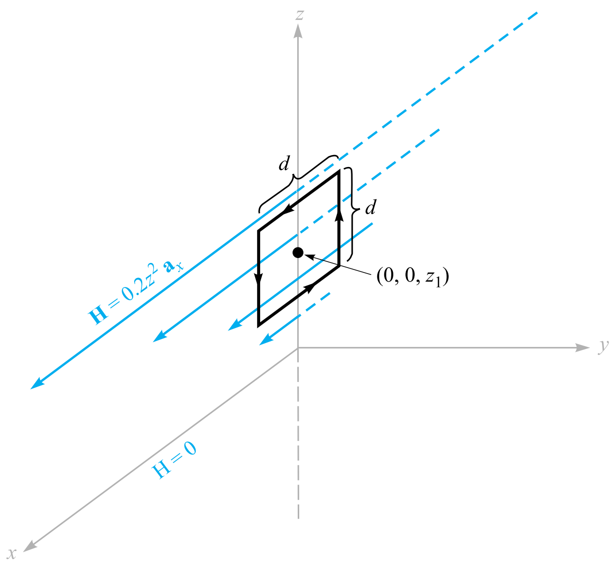

Figure 7.15 A square path of side d with its center on the z axis at $z = z_1$ is used to evaluate $\oint \mathbf{H} \cdot d\mathbf{L}$ and find curl $\mathbf{H}$.

Solution. We evaluate the line integral of $\mathbf{H}$ along the four segments, beginning at the top:

$$\oint \mathbf{H} \cdot d\mathbf{L} = 0.2\left(z_1 + \tfrac{1}{2}d\right)^2 d + 0 - 0.2\left(z_1 - \tfrac{1}{2}d\right)^2 d + 0$$
$$= 0.4 z_1 d^2$$

In the limit as the area approaches zero, we find

$$(\nabla \times \mathbf{H})_y = \lim_{d \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{d^2} = \lim_{d \to 0} \frac{0.4 z_1 d^2}{d^2} = 0.4 z_1$$

The other components are zero, so $\nabla \times \mathbf{H} = 0.4 z_1 \mathbf{a}_y$.

To evaluate the curl without trying to illustrate the definition or the evaluation of a line integral, we simply take the partial derivative indicated by (23):

$$\nabla \times \mathbf{H} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0.2z^2 & 0 & 0 \end{vmatrix} = \frac{\partial}{\partial z}(0.2z^2)\mathbf{a}_y = 0.4z\mathbf{a}_y$$

which checks with the preceding result when $z = z_1$.

Returning now to complete our original examination of the application of Ampère's circuital law to a differential-sized path, we may combine (18)–(20), (22), and (24),

$$\text{curl } \mathbf{H} = \nabla \times \mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\mathbf{a}_y$$
$$+ \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\mathbf{a}_z = \mathbf{J} \tag{27}$$

and write the *point form of Ampère's circuital law,*

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}} \tag{28}$$

This is the second of Maxwell's four equations as they apply to non-time-varying conditions. We may also write the third of these equations at this time; it is the point form of $\oint \mathbf{E} \cdot d\mathbf{L} = 0$, or

$$\boxed{\nabla \times \mathbf{E} = 0} \tag{29}$$

The fourth equation appears in Section 7.5.

D7.4. (*a*) Evaluate the closed line integral of $\mathbf{H}$ about the rectangular path $P_1(2, 3, 4)$ to $P_2(4, 3, 4)$ to $P_3(4, 3, 1)$ to $P_4(2, 3, 1)$ to P_1, given $\mathbf{H} = 3z\mathbf{a}_x - 2x^3\mathbf{a}_z$ A/m. (*b*) Determine the quotient of the closed line integral and the area enclosed by the path as an approximation to $(\nabla \times \mathbf{H})_y$. (*c*) Determine $(\nabla \times \mathbf{H})_y$ at the center of the area.

Ans. 354 A; 59 A/m^2; 57 A/m^2

D7.5. Calculate the value of the vector current density: (*a*) in rectangular coordinates at $P_A(2, 3, 4)$ if $\mathbf{H} = x^2z\mathbf{a}_y - y^2x\mathbf{a}_z$; (*b*) in cylindrical coordinates at $P_B(1.5, 90^\circ, 0.5)$ if $\mathbf{H} = \frac{2}{\rho}(\cos 0.2\phi)\mathbf{a}_\rho$; (*c*) in spherical coordinates at $P_C(2, 30^\circ, 20^\circ)$ if $\mathbf{H} = \frac{1}{\sin\theta}\mathbf{a}_\theta$.

Ans. $-16\mathbf{a}_x + 9\mathbf{a}_y + 16\mathbf{a}_z$ A/m^2; $0.055\mathbf{a}_z$ A/m^2; $\mathbf{a}_\phi$ A/m^2

7.4 STOKES' THEOREM

Although Section 7.3 was devoted primarily to a discussion of the curl operation, the contribution to the subject of magnetic fields should not be overlooked. From Ampère's circuital law we derived one of Maxwell's equations, $\nabla \times \mathbf{H} = \mathbf{J}$. This latter equation should be considered the point form of Ampère's circuital law and applies on a "per-unit-area" basis. In this section we shall again devote a major share of the material to the mathematical theorem known as Stokes' theorem, but in the process we will show that we may obtain Ampère's circuital law from $\nabla \times \mathbf{H} = \mathbf{J}$. In other words, we are then prepared to obtain the integral form from the point form or to obtain the point form from the integral form.

Consider the surface S of Figure 7.16, which is broken up into incremental surfaces of area ΔS. If we apply the definition of the curl to one of these incremental surfaces, then

$$\frac{\oint \mathbf{H} \cdot d\mathbf{L}_{\Delta S}}{\Delta S} \doteq (\nabla \times \mathbf{H})_N$$

where the N subscript again indicates the right-hand normal to the surface. The subscript on $d\mathbf{L}_{\Delta S}$ indicates that the closed path is the perimeter of an incremental area ΔS. This result may also be written

$$\frac{\oint \mathbf{H} \cdot d\mathbf{L}_{\Delta S}}{\Delta S} \doteq (\nabla \times \mathbf{H}) \cdot \mathbf{a}_N$$

or

$$\oint \mathbf{H} \cdot d\mathbf{L}_{\Delta S} \doteq (\nabla \times \mathbf{H}) \cdot \mathbf{a}_N \Delta S = (\nabla \times \mathbf{H}) \cdot \Delta\mathbf{S}$$

where $\mathbf{a}_N$ is a unit vector in the direction of the right-hand normal to ΔS.

Now let us determine this circulation for every ΔS comprising S and sum the results. As we evaluate the closed line integral for each ΔS, some cancellation will occur

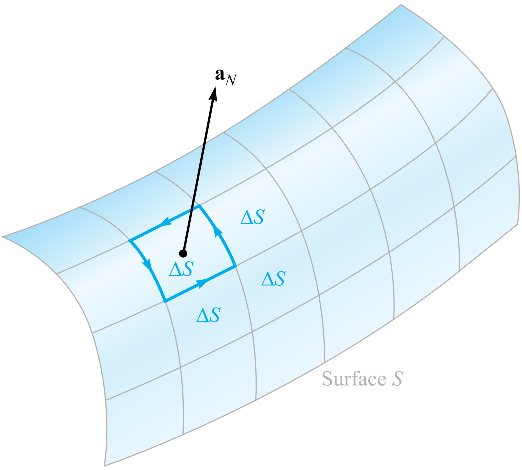

Figure 7.16 The sum of the closed line integrals about the perimeter of every ΔS is the same as the closed line integral about the perimeter of S because of cancellation on every interior path.

because every *interior* wall is covered once in each direction. The only boundaries on which cancellation cannot occur form the outside boundary, the path enclosing S. Therefore we have

$$\boxed{\oint \mathbf{H} \cdot d\mathbf{L} \equiv \int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S}} \qquad (30)$$

where $d\mathbf{L}$ is taken only on the perimeter of S.

Equation (30) is an identity, holding for any vector field, and is known as *Stokes' theorem*.

EXAMPLE 7.3

A numerical example may help to illustrate the geometry involved in Stokes' theorem. Consider the portion of a sphere shown in Figure 7.17. The surface is specified by $r = 4$, $0 \le \theta \le 0.1\pi$, $0 \le \phi \le 0.3\pi$, and the closed path forming its perimeter is composed of three circular arcs. We are given the field $\mathbf{H} = 6r \sin\phi \mathbf{a}_r + 18r \sin\theta \cos\phi \mathbf{a}_\phi$ and are asked to evaluate each side of Stokes' theorem.

Solution. The first path segment is described in spherical coordinates by $r = 4$, $0 \le \theta \le 0.1\pi$, $\phi = 0$; the second one by $r = 4$, $\theta = 0.1\pi$, $0 \le \phi \le 0.3\pi$; and the third by $r = 4$, $0 \le \theta \le 0.1\pi$, $\phi = 0.3\pi$. The differential path element $d\mathbf{L}$ is the vector sum of the three differential lengths of the spherical coordinate system first discussed in Section 1.9,

$$d\mathbf{L} = dr\,\mathbf{a}_r + r\,d\theta\,\mathbf{a}_\theta + r\sin\theta\,d\phi\,\mathbf{a}_\phi$$

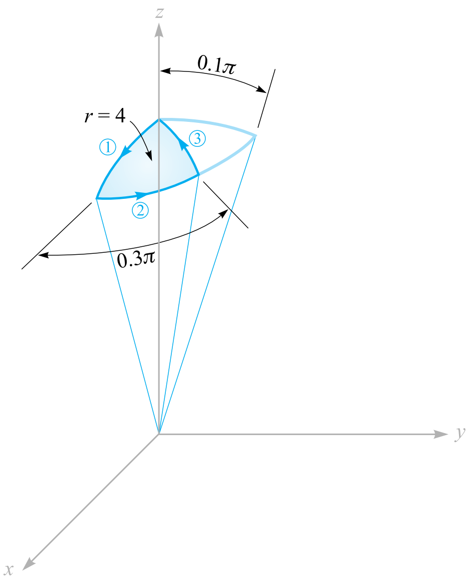

Figure 7.17 A portion of a spherical cap is used as a surface and a closed path to illustrate Stokes' theorem.

The first term is zero on all three segments of the path since $r = 4$ and $dr = 0$, the second is zero on segment 2 as θ is constant, and the third term is zero on both segments 1 and 3. Thus,

$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_1 H_\theta r\, d\theta + \int_2 H_\phi r \sin\theta\, d\phi + \int_3 H_\theta r\, d\theta$$

Because $H_\theta = 0$, we have only the second integral to evaluate,

$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_0^{0.3\pi} [18(4) \sin 0.1\pi \cos\phi] 4 \sin 0.1\pi\, d\phi$$
$$= 288 \sin^2 0.1\pi \sin 0.3\pi = 22.2 \text{ A}$$

We next attack the surface integral. First, we use (26) to find

$$\nabla \times \mathbf{H} = \frac{1}{r\sin\theta}(36r \sin\theta \cos\theta \cos\phi)\mathbf{a}_r + \frac{1}{r}\left(\frac{1}{\sin\theta} 6r \cos\phi - 36r \sin\theta \cos\phi\right)\mathbf{a}_\theta$$

Because $d\mathbf{S} = r^2 \sin\theta\, d\theta\, d\phi\, \mathbf{a}_r$, the integral is

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = \int_0^{0.3\pi} \int_0^{0.1\pi} (36 \cos\theta \cos\phi) 16 \sin\theta\, d\theta\, d\phi$$
$$= \int_0^{0.3\pi} 576 \left(\tfrac{1}{2} \sin^2\theta\right) \Big|_0^{0.1\pi} \cos\phi\, d\phi$$
$$= 288 \sin^2 0.1\pi \sin 0.3\pi = 22.2 \text{ A}$$

Thus, the results check Stokes' theorem, and we note in passing that a current of 22.2 A is flowing upward through this section of a spherical cap.

Next, let us see how easy it is to obtain Ampère's circuital law from $\nabla \times \mathbf{H} = \mathbf{J}$. We merely have to dot each side by $d\mathbf{S}$, integrate each side over the same (open) surface S, and apply Stokes' theorem:

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = \int_S \mathbf{J} \cdot d\mathbf{S} = \oint \mathbf{H} \cdot d\mathbf{L}$$

The integral of the current density over the surface S is the total current I passing through the surface, and therefore

$$\oint \mathbf{H} \cdot d\mathbf{L} = I$$

This short derivation shows clearly that the current I, described as being "enclosed by the closed path," is also the current passing through any of the infinite number of surfaces that have the closed path as a perimeter.

Stokes' theorem relates a surface integral to a closed line integral. It should be recalled that the divergence theorem relates a volume integral to a closed surface integral. Both theorems find their greatest use in general vector proofs. As an example, let us find another expression for $\nabla \cdot \nabla \times \mathbf{A}$, where $\mathbf{A}$ represents any vector field. The result must be a scalar (why?), and we may let this scalar be T, or

$$\nabla \cdot \nabla \times \mathbf{A} = T$$

Multiplying by dv and integrating throughout any volume v,

$$\int_{\text{vol}} (\nabla \cdot \nabla \times \mathbf{A})\, dv = \int_{\text{vol}} T\, dv$$

we first apply the divergence theorem to the left side, obtaining

$$\oint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \int_{\text{vol}} T\, dv$$

The left side is the surface integral of the curl of $\mathbf{A}$ over the *closed* surface surrounding the volume v. Stokes' theorem relates the surface integral of the curl of $\mathbf{A}$ over the *open* surface enclosed by a given closed path. If we think of the path as the opening of a laundry bag and the open surface as the surface of the bag itself, we see that as we gradually approach a closed surface by pulling on the drawstrings, the closed path becomes smaller and smaller and finally disappears as the surface becomes closed. Hence, the application of Stokes' theorem to a *closed* surface produces a zero result, and we have

$$\int_{\text{vol}} T\, dv = 0$$

Because this is true for any volume, it is true for the differential volume dv,

$$T\,dv = 0$$

and therefore

$$T = 0$$

or

$$\boxed{\nabla \cdot \nabla \times \mathbf{A} \equiv 0} \tag{31}$$

Equation (31) is a useful identity of vector calculus.[6] Of course, it may also be proven easily by direct expansion in rectangular coordinates.

Let us apply the identity to the non-time-varying magnetic field for which

$$\nabla \times \mathbf{H} = \mathbf{J}$$

This shows quickly that

$$\nabla \cdot \mathbf{J} = 0$$

which is the same result we obtained earlier in the chapter by using the continuity equation.

Before introducing several new magnetic field quantities in the following section, we may review our accomplishments at this point. We initially accepted the Biot-Savart law as an experimental result,

$$\mathbf{H} = \oint \frac{I\,d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$$

and tentatively accepted Ampère's circuital law, subject to later proof,

$$\oint \mathbf{H} \cdot d\mathbf{L} = I$$

From Ampère's circuital law the definition of curl led to the point form of this same law,

$$\nabla \times \mathbf{H} = \mathbf{J}$$

We now see that Stokes' theorem enables us to obtain the integral form of Ampère's circuital law from the point form.

D7.6. Evaluate both sides of Stokes' theorem for the field $\mathbf{H} = 6xy\mathbf{a}_x - 3y^2\mathbf{a}_y$ A/m and the rectangular path around the region, $2 \le x \le 5$, $-1 \le y \le 1$, $z = 0$. Let the positive direction of $d\mathbf{S}$ be $\mathbf{a}_z$.

Ans. -126 A; -126 A

[6] This and other vector identities are tabulated in Appendix A.3.

7.5 MAGNETIC FLUX AND MAGNETIC FLUX DENSITY

In free space, let us define the *magnetic flux density* **B** as

$$\boxed{\mathbf{B} = \mu_0 \mathbf{H}} \qquad \text{(free space only)} \tag{32}$$

where **B** is measured in webers per square meter (Wb/m^2) or in a newer unit adopted in the International System of Units, tesla (T). An older unit that is often used for magnetic flux density is the gauss (G), where 1 T or 1Wb/m^2 is the same as 10, 000 G. The constant μ_0 is not dimensionless and has the *defined value* for free space, in henrys per meter (H/m), of

$$\boxed{\mu_0 = 4\pi \times 10^{-7} \text{ H/m}} \tag{33}$$

The name given to μ_0 is the *permeability* of free space.

We should note that since **H** is measured in amperes per meter, the weber is dimensionally equal to the product of henrys and amperes. Considering the henry as a new unit, the weber is merely a convenient abbreviation for the product of henrys and amperes. When time-varying fields are introduced, it will be shown that a weber is also equivalent to the product of volts and seconds.

The magnetic-flux-density vector **B**, as the name weber per square meter implies, is a member of the flux-density family of vector fields. One of the possible analogies between electric and magnetic fields[7] compares the laws of Biot-Savart and Coulomb, thus establishing an analogy between **H** and **E**. The relations $\mathbf{B} = \mu_0 \mathbf{H}$ and $\mathbf{D} = \epsilon_0 \mathbf{E}$ then lead to an analogy between **B** and **D**. If **B** is measured in teslas or webers per square meter, then magnetic flux should be measured in webers. Let us represent magnetic flux by Φ and define Φ as the flux passing through any designated area,

$$\boxed{\Phi = \int_S \mathbf{B} \cdot d\mathbf{S} \text{ Wb}} \tag{34}$$

Our analogy should now remind us of the electric flux Ψ, measured in coulombs, and of Gauss's law, which states that the total flux passing through any closed surface is equal to the charge enclosed,

$$\Psi = \oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

The charge Q is the source of the lines of electric flux and these lines begin and terminate on positive and negative charges, respectively.

[7] An alternate analogy is presented in Section 9.2.

No such source has ever been discovered for the lines of magnetic flux. In the example of the infinitely long straight filament carrying a direct current I, the $\mathbf{H}$ field formed concentric circles about the filament. Because $\mathbf{B} = \mu_0 \mathbf{H}$, the $\mathbf{B}$ field is of the same form. The magnetic flux lines are closed and do not terminate on a "magnetic charge." For this reason Gauss's law for the magnetic field is

$$\boxed{\oint_S \mathbf{B} \cdot d\mathbf{S} = 0} \tag{35}$$

and application of the divergence theorem shows us that

$$\boxed{\nabla \cdot \mathbf{B} = 0} \tag{36}$$

Equation (36) is the last of Maxwell's four equations as they apply to static electric fields and steady magnetic fields. Collecting these equations, we then have for static electric fields and steady magnetic fields

$$\boxed{\begin{aligned} \nabla \cdot \mathbf{D} &= \rho_v \\ \nabla \times \mathbf{E} &= 0 \\ \nabla \times \mathbf{H} &= \mathbf{J} \\ \nabla \cdot \mathbf{B} &= 0 \end{aligned}} \tag{37}$$

To these equations we may add the two expressions relating $\mathbf{D}$ to $\mathbf{E}$ and $\mathbf{B}$ to $\mathbf{H}$ in free space,

$$\boxed{\mathbf{D} = \epsilon_0 \mathbf{E}} \tag{38}$$

$$\boxed{\mathbf{B} = \mu_0 \mathbf{H}} \tag{39}$$

We have also found it helpful to define an electrostatic potential,

$$\boxed{\mathbf{E} = -\nabla V} \tag{40}$$

and we will discuss a potential for the steady magnetic field in the following section. In addition, we extended our coverage of electric fields to include conducting materials and dielectrics, and we introduced the polarization **P.** A similar treatment will be applied to magnetic fields in the next chapter.

Returning to (37), it may be noted that these four equations specify the divergence and curl of an electric and a magnetic field. The corresponding set of four integral equations that apply to static electric fields and steady magnetic fields is

$$
\begin{aligned}
&\oint_S \mathbf{D}\cdot d\mathbf{S} = Q = \int_{\text{vol}} \rho_\nu d\nu \\
&\oint \mathbf{E}\cdot d\mathbf{L} = 0 \\
&\oint \mathbf{H}\cdot d\mathbf{L} = I = \int_S \mathbf{J}\cdot d\mathbf{S} \\
&\oint_S \mathbf{B}\cdot d\mathbf{S} = 0
\end{aligned}
\tag{41}
$$

Our study of electric and magnetic fields would have been much simpler if we could have begun with either set of equations, (37) or (41). With a good knowledge of vector analysis, such as we should now have, either set may be readily obtained from the other by applying the divergence theorem or Stokes' theorem. The various experimental laws can be obtained easily from these equations.

As an example of the use of flux and flux density in magnetic fields, let us find the flux between the conductors of the coaxial line of Figure 7.8a. The magnetic field intensity was found to be

$$H_\phi = \frac{I}{2\pi\rho} \qquad (a < \rho < b)$$

and therefore

$$\mathbf{B} = \mu_0\mathbf{H} = \frac{\mu_0 I}{2\pi\rho}\mathbf{a}_\phi$$

The magnetic flux contained between the conductors in a length d is the flux crossing any radial plane extending from $\rho = a$ to $\rho = b$ and from, say, $z = 0$ to $z = d$

$$\Phi = \int_S \mathbf{B}\cdot d\mathbf{S} = \int_0^d \int_a^b \frac{\mu_0 I}{2\pi\rho}\mathbf{a}_\phi \cdot d\rho\, dz\, \mathbf{a}_\phi$$

or

$$\Phi = \frac{\mu_0 I d}{2\pi} \ln\frac{b}{a} \tag{42}$$

This expression will be used later to obtain the inductance of the coaxial transmission line.

D7.7. A solid conductor of circular cross section is made of a homogeneous nonmagnetic material. If the radius $a = 1$ mm, the conductor axis lies on the z axis, and the total current in the $\mathbf{a}_z$ direction is 20 A, find: (a) H_ϕ at $\rho = 0.5$ mm; (b) B_ϕ at $\rho = 0.8$ mm; (c) the total magnetic flux per unit length inside the conductor; (d) the total flux for $\rho < 0.5$ mm; (e) the total magnetic flux outside the conductor.

Ans. 1592 A/m; 3.2 mT; 2 μWb/m; 0.5 μWb; ∞

7.6 THE SCALAR AND VECTOR MAGNETIC POTENTIALS

The solution of electrostatic field problems is greatly simplified by the use of the scalar electrostatic potential V. Although this potential possesses a very real physical significance for us, it is mathematically no more than a stepping-stone which allows us to solve a problem by several smaller steps. Given a charge configuration, we may first find the potential and then from it the electric field intensity.

We should question whether or not such assistance is available in magnetic fields. Can we define a potential function which may be found from the current distribution and from which the magnetic fields may be easily determined? Can a scalar magnetic potential be defined, similar to the scalar electrostatic potential? We will show in the next few pages that the answer to the first question is yes, but the second must be answered "sometimes." Let us attack the second question first by assuming the existence of a scalar magnetic potential, which we designate V_m, whose negative gradient gives the magnetic field intensity

$$\mathbf{H} = -\nabla V_m$$

The selection of the negative gradient provides a closer analogy to the electric potential and to problems which we have already solved.

This definition must not conflict with our previous results for the magnetic field, and therefore

$$\nabla \times \mathbf{H} = \mathbf{J} = \nabla \times (-\nabla V_m)$$

However, the curl of the gradient of any scalar is identically zero, a vector identity the proof of which is left for a leisure moment. Therefore, we see that if $\mathbf{H}$ is to be defined as the gradient of a scalar magnetic potential, then current density must be zero throughout the region in which the scalar magnetic potential is so defined. We then have

$$\boxed{\mathbf{H} = -\nabla V_m \quad (\mathbf{J} = 0)} \tag{43}$$

Because many magnetic problems involve geometries in which the current-carrying conductors occupy a relatively small fraction of the total region of interest, it is evident that a scalar magnetic potential can be useful. The scalar magnetic potential is also applicable in the case of permanent magnets. The dimensions of V_m are obviously amperes.

This scalar potential also satisfies Laplace's equation. In free space,

$$\nabla \cdot \mathbf{B} = \mu_0 \nabla \cdot \mathbf{H} = 0$$

and hence

$$\mu_0 \nabla \cdot (-\nabla V_m) = 0$$

or

$$\boxed{\nabla^2 V_m = 0 \quad (\mathbf{J} = 0)} \tag{44}$$

We will see later that V_m continues to satisfy Laplace's equation in homogeneous magnetic materials; it is not defined in any region in which current density is present.

Although we shall consider the scalar magnetic potential to a much greater extent in Chapter 8, when we introduce magnetic materials and discuss the magnetic circuit, one difference between V and V_m should be pointed out now: V_m is not a single-valued function of position. The electric potential V is single-valued; once a zero reference is assigned, there is only one value of V associated with each point in space. Such is not the case with V_m. Consider the cross section of the coaxial line shown in Figure 7.18. In the region $a < \rho < b$, $\mathbf{J} = 0$, and we may establish a scalar magnetic potential. The value of $\mathbf{H}$ is

$$\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_\phi$$

where I is the total current flowing in the $\mathbf{a}_z$ direction in the inner conductor. We find V_m by integrating the appropriate component of the gradient. Applying (43),

$$\frac{I}{2\pi\rho} = -\nabla V_m \Big|_\phi = -\frac{1}{\rho}\frac{\partial V_m}{\partial \phi}$$

or

$$\frac{\partial V_m}{\partial \phi} = -\frac{I}{2\pi}$$

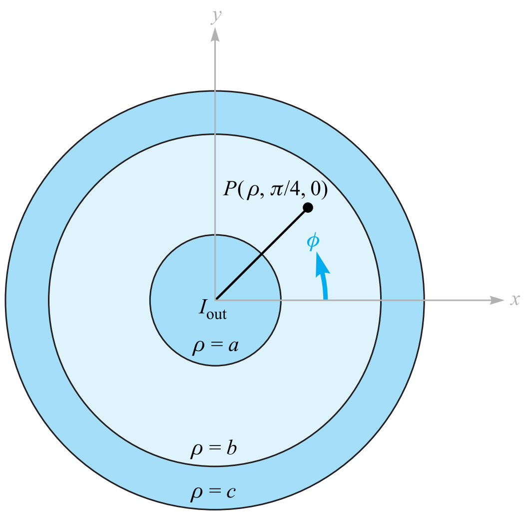

Figure 7.18 The scalar magnetic potential V_m is a multivalued function of ϕ in the region $a < \rho < b$. The electrostatic potential is always single valued.

Thus,

$$V_m = -\frac{I}{2\pi}\phi$$

where the constant of integration has been set equal to zero. What value of potential do we associate with point P, where $\phi = \pi/4$? If we let V_m be zero at $\phi = 0$ and proceed counterclockwise around the circle, the magnetic potential goes negative linearly. When we have made one circuit, the potential is $-I$, but that was the point at which we said the potential was zero a moment ago. At P, then, $\phi = \pi/4$, $9\pi/4$, $17\pi/4, \ldots,$ or $-7\pi/4$, $-15\pi/4$, $-23\pi/4, \ldots,$ or

$$V_{mP} = \frac{I}{2\pi}\left(2n - \tfrac{1}{4}\right)\pi \quad (n = 0, \pm 1, \pm 2, \ldots)$$

or

$$V_{mP} = I\left(n - \tfrac{1}{8}\right) \quad (n = 0, \pm 1, \pm 2, \ldots)$$

The reason for this multivaluedness may be shown by a comparison with the electrostatic case. There, we know that

$$\nabla \times \mathbf{E} = 0$$

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

and therefore the line integral

$$V_{ab} = -\int_b^a \mathbf{E} \cdot d\mathbf{L}$$

is independent of the path. In the magnetostatic case, however,

$$\nabla \times \mathbf{H} = 0 \quad (\text{wherever } \mathbf{J} = 0)$$

but

$$\oint \mathbf{H} \cdot d\mathbf{L} = I$$

even if $\mathbf{J}$ is zero along the path of integration. Every time we make another complete lap around the current, the result of the integration increases by I. If no current I is enclosed by the path, then a single-valued potential function may be defined. In general, however,

$$\boxed{V_{m,ab} = -\int_b^a \mathbf{H} \cdot d\mathbf{L} \quad (\text{specified path})} \tag{45}$$

where a specific path or type of path must be selected. We should remember that the electrostatic potential V is a conservative field; the magnetic scalar potential V_m is not a conservative field. In our coaxial problem, let us erect a barrier[8] at $\phi = \pi$; we

[8] This corresponds to the more precise mathematical term "branch cut."

agree not to select a path that crosses this plane. Therefore, we cannot encircle I, and a single-valued potential is possible. The result is seen to be

$$V_m = -\frac{I}{2\pi}\phi \quad (-\pi < \phi < \pi)$$

and

$$V_{mP} = -\frac{I}{8} \quad \left(\phi = \frac{\pi}{4}\right)$$

The scalar magnetic potential is evidently the quantity whose equipotential surfaces will form curvilinear squares with the streamlines of **H** in Figure 7.4. This is one more facet of the analogy between electric and magnetic fields about which we will have more to say in the next chapter.

Let us temporarily leave the scalar magnetic potential now and investigate a vector magnetic potential. This vector field is one which is extremely useful in studying radiation from antennas (as we will find in Chapter 14) as well as radiation leakage from transmission lines, waveguides, and microwave ovens. The vector magnetic potential may be used in regions where the current density is zero or nonzero, and we shall also be able to extend it to the time-varying case later.

Our choice of a vector magnetic potential is indicated by noting that

$$\nabla \cdot \mathbf{B} = 0$$

Next, a vector identity that we proved in Section 7.4 shows that the divergence of the curl of any vector field is zero. Therefore, we select

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}} \tag{46}$$

where A signifies a *vector magnetic potential,* and we automatically satisfy the condition that the magnetic flux density shall have zero divergence. The **H** field is

$$\mathbf{H} = \frac{1}{\mu_0}\nabla \times \mathbf{A}$$

and

$$\nabla \times \mathbf{H} = \mathbf{J} = \frac{1}{\mu_0}\nabla \times \nabla \times \mathbf{A}$$

The curl of the curl of a vector field is not zero and is given by a fairly complicated expression,[9] which we need not know now in general form. In specific cases for which the form of **A** is known, the curl operation may be applied twice to determine the current density.

[9] $\nabla \times \nabla \times \mathrm{A} \equiv \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathrm{A}$. In rectangular coordinates, it may be shown that $\nabla^2\mathrm{A} \equiv \nabla^2\mathrm{A}_x\mathbf{a}_x + \nabla^2\mathrm{A}_y\mathbf{a}_y + \nabla^2\mathrm{A}_z\mathbf{a}_z$. In other coordinate systems, $\nabla^2\mathrm{A}$ may be found by evaluating the second-order partial derivatives in $\nabla^2\mathrm{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathrm{A}$.

Equation (46) serves as a useful definition of the *vector magnetic potential* **A**. Because the curl operation implies differentiation with respect to a length, the units of **A** are webers per meter.

As yet we have seen only that the definition for **A** does not conflict with any previous results. It still remains to show that this particular definition can help us to determine magnetic fields more easily. We certainly cannot identify **A** with any easily measured quantity or history-making experiment.

We will show in Section 7.7 that, given the Biot-Savart law, the definition of **B**, and the definition of **A**, **A** may be determined from the differential current elements by

$$\boxed{\mathbf{A} = \oint \frac{\mu_0 I\, d\mathbf{L}}{4\pi R}} \tag{47}$$

The significance of the terms in (47) is the same as in the Biot-Savart law; a direct current I flows along a filamentary conductor of which any differential length $d\mathbf{L}$ is distant R from the point at which **A** is to be found. Because we have defined **A** only through specification of its curl, it is possible to add the gradient of any scalar field to (47) without changing **B** or **H**, for the curl of the gradient is identically zero. In steady magnetic fields, it is customary to set this possible added term equal to zero.

The fact that **A** is a vector magnetic *potential* is more apparent when (47) is compared with the similar expression for the electrostatic potential,

$$V = \int \frac{\rho_L dL}{4\pi\epsilon_0 R}$$

Each expression is the integral along a line source, in one case line charge and in the other case line current; each integrand is inversely proportional to the distance from the source to the point of interest; and each involves a characteristic of the medium (here free space), the permeability or the permittivity.

Equation (47) may be written in differential form,

$$d\mathbf{A} = \frac{\mu_0 I\, d\mathbf{L}}{4\pi R} \tag{48}$$

if we again agree not to attribute any physical significance to any magnetic fields we obtain from (48) until the *entire closed path in which the current flows is considered*.

With this reservation, let us go right ahead and consider the vector magnetic potential field about a differential filament. We locate the filament at the origin in free space, as shown in Figure 7.19, and allow it to extend in the positive z direction so that $d\mathbf{L} = dz\,\mathbf{a}_z$. We use cylindrical coordinates to find $d\mathbf{A}$ at the point (ρ, ϕ, z):

$$d\mathbf{A} = \frac{\mu_0 I\, dz\,\mathbf{a}_z}{4\pi\sqrt{\rho^2 + z^2}}$$

or

$$d\mathbf{A}_z = \frac{\mu_0 I\, dz}{4\pi\sqrt{\rho^2 + z^2}} \qquad dA_\phi = 0 \quad dA_\rho = 0 \tag{49}$$

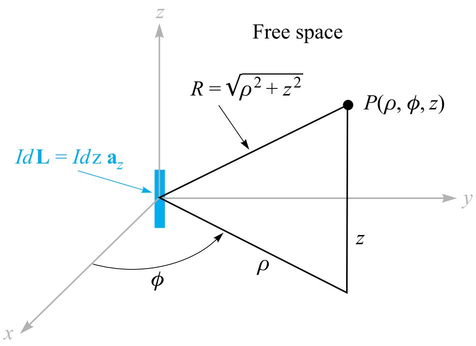

Figure 7.19 The differential current element $I\, dz\mathbf{a}_z$ at the origin establishes the differential vector magnetic potential field, $d\mathbf{A} = \dfrac{\mu_0 I\, dz\mathbf{a}_z}{4\pi\sqrt{\rho^2 + z^2}}$ at $P(\rho, \phi, z)$.

We note that the direction of $d\mathbf{A}$ is the same as that of $I\, d\mathbf{L}$. Each small section of a current-carrying conductor produces a contribution to the total vector magnetic potential which is in the same direction as the current flow in the conductor. The magnitude of the vector magnetic potential varies inversely with the distance to the current element, being strongest in the neighborhood of the current and gradually falling off to zero at distant points. Skilling[10] describes the vector magnetic potential field as "like the current distribution but fuzzy around the edges, or like a picture of the current out of focus."

In order to find the magnetic field intensity, we must take the curl of (49) in cylindrical coordinates, leading to

$$d\mathbf{H} = \frac{1}{\mu_0}\nabla \times d\mathbf{A} = \frac{1}{\mu_0}\left(-\frac{\partial dA_z}{\partial \rho}\right)\mathbf{a}_\phi$$

or

$$d\mathbf{H} = \frac{I\, dz}{4\pi}\frac{\rho}{(\rho^2 + z^2)^{3/2}}\mathbf{a}_\phi$$

which is easily shown to be the same as the value given by the Biot-Savart law.

Expressions for the vector magnetic potential $\mathbf{A}$ can also be obtained for a current source which is distributed. For a current sheet $\mathbf{K}$, the differential current element becomes

$$I\, d\mathbf{L} = \mathbf{K}\, dS$$

In the case of current flow throughout a volume with a density $\mathbf{J}$, we have

$$I\, d\mathbf{L} = \mathbf{J}\, dv$$

[10] See the References at the end of the chapter.

In each of these two expressions the vector character is given to the current. For the filamentary element it is customary, although not necessary, to use $I\,d\mathbf{L}$ instead of $\mathbf{I}\,dL$. Since the magnitude of the filamentary element is constant, we have chosen the form which allows us to remove one quantity from the integral. The alternative expressions for **A** are then

$$\boxed{\mathbf{A} = \int_S \frac{\mu_0 \mathbf{K}\, dS}{4\pi R}} \tag{50}$$

and

$$\boxed{\mathbf{A} = \int_{\text{vol}} \frac{\mu_0 \mathbf{J}\, dv}{4\pi R}} \tag{51}$$

Equations (47), (50), and (51) express the vector magnetic potential as an integration over all of its sources. From a comparison of the form of these integrals with those which yield the electrostatic potential, it is evident that once again the zero reference for **A** is at infinity, for no finite current element can produce any contribution as $R \to \infty$. We should remember that we very seldom used the similar expressions for V; too often our theoretical problems included charge distributions that extended to infinity, and the result would be an infinite potential everywhere. Actually, we calculated very few potential fields until the differential form of the potential equation was obtained, $\nabla^2 V = -\rho_v/\epsilon$, or better yet, $\nabla^2 V = 0$. We were then at liberty to select our own zero reference.

The analogous expressions for **A** will be derived in the next section, and an example of the calculation of a vector magnetic potential field will be completed.

D7.8. A current sheet, $\mathbf{K} = 2.4\mathbf{a}_z$ A/m, is present at the surface $\rho = 1.2$ in free space. (a) Find **H** for $\rho > 1.2$. Find V_m at $P(\rho = 1.5, \phi = 0.6\pi, z = 1)$ if: (b) $V_m = 0$ at $\phi = 0$ and there is a barrier at $\phi = \pi$; (c) $V_m = 0$ at $\phi = 0$ and there is a barrier at $\phi = \pi/2$; (d) $V_m = 0$ at $\phi = \pi$ and there is a barrier at $\phi = 0$; (e) $V_m = 5$ V at $\phi = \pi$ and there is a barrier at $\phi = 0.8\pi$.

Ans. $\dfrac{2.88}{\rho}\mathbf{a}_\phi$; -5.43 V; 12.7 V; 3.62 V; -9.48 V

D7.9. The value of **A** within a solid nonmagnetic conductor of radius a carrying a total current I in the $\mathbf{a}_z$ direction may be found easily. Using the known value of **H** or **B** for $\rho < a$, then (46) may be solved for **A**. Select $A = (\mu_0 I \ln 5)/2\pi$ at $\rho = a$ (to correspond with an example in the next section) and find **A** at $\rho =$: (a) 0; (b) $0.25a$; (c) $0.75a$; (d) a.

Ans. $0.422I\mathbf{a}_z$ μWb/m; $0.416I\mathbf{a}_z$ μWb/m; $0.366I\mathbf{a}_z$ μWb/m; $0.322I\mathbf{a}_z$ μWb/m

7.7 DERIVATION OF THE STEADY-MAGNETIC-FIELD LAWS

We will now supply the promised proofs of the several relationships between the magnetic field quantities. All these relationships may be obtained from the definitions of **H**,

$$\mathbf{H} = \oint \frac{I\,d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} \tag{3}$$

of **B** (in free space),

$$\mathbf{B} = \mu_0 \mathbf{H} \tag{32}$$

and of **A**,

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{46}$$

Let us first assume that we may express **A** by the last equation of Section 7.6,

$$\mathbf{A} = \int_{\text{vol}} \frac{\mu_0 \mathbf{J}\,dv}{4\pi R} \tag{51}$$

and then demonstrate the correctness of (51) by showing that (3) follows. First, we should add subscripts to indicate the point at which the current element is located (x_1, y_1, z_1) and the point at which **A** is given (x_2, y_2, z_2). The differential volume element dv is then written dv_1 and in rectangular coordinates would be $dx_1\,dy_1\,dz_1$. The variables of integration are x_1, y_1, and z_1. Using these subscripts, then,

$$\mathbf{A}_2 = \int_{\text{vol}} \frac{\mu_0 \mathbf{J}_1 dv_1}{4\pi R_{12}} \tag{52}$$

From (32) and (46) we have

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} = \frac{\nabla \times \mathbf{A}}{\mu_0} \tag{53}$$

To show that (3) follows from (52), it is necessary to substitute (52) into (53). This step involves taking the curl of $\mathbf{A}_2$, a quantity expressed in terms of the variables x_2, y_2, and z_2, and the curl therefore involves partial derivatives with respect to x_2, y_2, and z_2. We do this, placing a subscript on the del operator to remind us of the variables involved in the partial differentiation process,

$$\mathbf{H}_2 = \frac{\nabla_2 \times \mathbf{A}_2}{\mu_0} = \frac{1}{\mu_0} \nabla_2 \times \int_{\text{vol}} \frac{\mu_0 \mathbf{J}_1 dv_1}{4\pi R_{12}}$$

The order of partial differentiation and integration is immaterial, and $\mu_0/4\pi$ is constant, allowing us to write

$$\mathbf{H}_2 = \frac{1}{4\pi} \int_{\text{vol}} \nabla_2 \times \frac{\mathbf{J}_1 dv_1}{R_{12}}$$

The curl operation within the integrand represents partial differentiation with respect to x_2, y_2, and z_2. The differential volume element dv_1 is a scalar and a function only of x_1, y_1, and z_1. Consequently, it may be factored out of the curl operation as any other constant, leaving

$$\mathbf{H}_2 = \frac{1}{4\pi} \int_{\text{vol}} \left(\nabla_2 \times \frac{\mathbf{J}_1}{R_{12}} \right) d\nu_1 \tag{54}$$

The curl of the product of a scalar and a vector is given by an identity which may be checked by expansion in rectangular coordinates or obtained from Appendix A.3,

$$\nabla \times (S\mathbf{V}) \equiv (\nabla S) \times \mathbf{V} + S(\nabla \times \mathbf{V}) \tag{55}$$

This identity is used to expand the integrand of (54),

$$\mathbf{H}_2 = \frac{1}{4\pi} \int_{\text{vol}} \left[\left(\nabla_2 \frac{1}{R_{12}} \right) \times \mathbf{J}_1 + \frac{1}{R_{12}} (\nabla_2 \times \mathbf{J}_1) \right] d\nu_1 \tag{56}$$

The second term of this integrand is zero because $\nabla_2 \times \mathbf{J}_1$ indicates partial derivatives of a function of x_1, y_1, and z_1, taken with respect to the variables x_2, y_2, and z_2; the first set of variables is not a function of the second set, and all partial derivatives are zero.

The first term of the integrand may be determined by expressing R_{12} in terms of the coordinate values,

$$R_{12} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

and taking the gradient of its reciprocal. Problem 7.42 shows that the result is

$$\nabla_2 \frac{1}{R_{12}} = -\frac{\mathbf{R}_{12}}{R_{12}^3} = -\frac{\mathbf{a}_{R12}}{R_{12}^2}$$

Substituting this result into (56), we have

$$\mathbf{H}_2 = -\frac{1}{4\pi} \int_{\text{vol}} \frac{\mathbf{a}_{R12} \times \mathbf{J}_1}{R_{12}^2} d\nu_1$$

or

$$\mathbf{H}_2 = \int_{\text{vol}} \frac{\mathbf{J}_1 \times \mathbf{a}_{R12}}{4\pi R_{12}^2} d\nu_1$$

which is the equivalent of (3) in terms of current density. Replacing $\mathbf{J}_1\, d\nu_1$ by $I_1\, d\mathbf{L}_1$, we may rewrite the volume integral as a closed line integral,

$$\mathbf{H}_2 = \oint \frac{I_1 d\mathbf{L}_1 \times \mathbf{a}_{R12}}{4\pi R_{12}^2}$$

Equation (51) is therefore correct and agrees with the three definitions (3), (32), and (46).

Next we will prove Ampère's circuital law in point form,

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{28}$$

Combining (28), (32), and (46), we obtain

$$\nabla \times \mathbf{H} = \nabla \times \frac{\mathbf{B}}{\mu_0} = \frac{1}{\mu_0} \nabla \times \nabla \times \mathbf{A} \tag{57}$$

We now need the expansion in rectangular coordinates for $\nabla \times \nabla \times \mathbf{A}$. Performing the indicated partial differentiations and collecting the resulting terms, we may write the result as

$$\boxed{\nabla \times \nabla \times \mathbf{A} \equiv \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}} \tag{58}$$

where

$$\boxed{\nabla^2 \mathbf{A} \equiv \nabla^2 A_x \mathbf{a}_x + \nabla^2 A_y \mathbf{a}_y + \nabla^2 A_z \mathbf{a}_z} \tag{59}$$

Equation (59) is the definition (in rectangular coordinates) of the *Laplacian of a vector*.

Substituting (58) into (57), we have

$$\nabla \times \mathbf{H} = \frac{1}{\mu_0}[\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}] \tag{60}$$

and now require expressions for the divergence and the Laplacian of $\mathbf{A}$.

We may find the divergence of $\mathbf{A}$ by applying the divergence operation to (52),

$$\nabla_2 \cdot \mathbf{A}_2 = \frac{\mu_0}{4\pi} \int_{\text{vol}} \nabla_2 \cdot \frac{\mathbf{J}_1}{R_{12}} dv_1 \tag{61}$$

and using the vector identity (44) of Section 4.8,

$$\nabla \cdot (S\mathbf{V}) \equiv \mathbf{V} \cdot (\nabla S) + S(\nabla \cdot \mathbf{V})$$

Thus,

$$\nabla_2 \cdot \mathbf{A}_2 = \frac{\mu_0}{4\pi} \int_{\text{vol}} \left[\mathbf{J}_1 \cdot \left(\nabla_2 \frac{1}{R_{12}} \right) + \frac{1}{R_{12}} (\nabla_2 \cdot \mathbf{J}_1) \right] dv_1 \tag{62}$$

The second part of the integrand is zero because $\mathbf{J}_1$ is not a function of x_2, y_2, and z_2.

We have already used the result that $\nabla_2(1/R_{12}) = -\mathbf{R}_{12}/R_{12}^3$, and it is just as easily shown that

$$\nabla_1 \frac{1}{R_{12}} = \frac{\mathbf{R}_{12}}{R_{12}^3}$$

or that

$$\nabla_1 \frac{1}{R_{12}} = -\nabla_2 \frac{1}{R_{12}}$$

Equation (62) can therefore be written as

$$\nabla_2 \cdot \mathbf{A}_2 = \frac{\mu_0}{4\pi} \int_{\text{vol}} \left[-\mathbf{J}_1 \cdot \left(\nabla_1 \frac{1}{R_{12}} \right) \right] dv_1$$

and the vector identity applied again,

$$\nabla_2 \cdot \mathbf{A}_2 = \frac{\mu_0}{4\pi} \int_{\text{vol}} \left[\frac{1}{R_{12}} (\nabla_1 \cdot \mathbf{J}_1) - \nabla_1 \cdot \left(\frac{\mathbf{J}_1}{R_{12}} \right) \right] dv_1 \tag{63}$$

Because we are concerned only with steady magnetic fields, the continuity equation shows that the first term of (63) is zero. Application of the divergence theorem to the second term gives

$$\nabla_2 \cdot \mathbf{A}_2 = -\frac{\mu_0}{4\pi} \oint_{S_1} \frac{\mathbf{J}_1}{R_{12}} \cdot d\mathbf{S}_1$$

where the surface S_1 encloses the volume throughout which we are integrating. This volume must include all the current, for the original integral expression for **A** was an integration such as to include the effect of all the current. Because there is no current outside this volume (otherwise we should have had to increase the volume to include it), we may integrate over a slightly larger volume or a slightly larger enclosing surface without changing **A.** On this larger surface the current density $\mathbf{J}_1$ must be zero, and therefore the closed surface integral is zero, since the integrand is zero. Hence the divergence of **A** is zero.

In order to find the Laplacian of the vector **A**, let us compare the x component of (51) with the similar expression for electrostatic potential,

$$A_x = \int_{\text{vol}} \frac{\mu_0 J_x d\nu}{4\pi R} \qquad V = \int_{\text{vol}} \frac{\rho_\nu \, d\nu}{4\pi\epsilon_0 R}$$

We note that one expression can be obtained from the other by a straightforward change of variable, J_x for ρ_ν, μ_0 for $1/\epsilon_0$, and A_x for V. However, we have derived some additional information about the electrostatic potential which we shall not have to repeat now for the x component of the vector magnetic potential. This takes the form of Poisson's equation,

$$\nabla^2 V = -\frac{\rho_\nu}{\epsilon_0}$$

which becomes, after the change of variables,

$$\nabla^2 A_x = -\mu_0 J_x$$

Similarly, we have

$$\nabla^2 A_y = -\mu_0 J_y$$

and

$$\nabla^2 A_z = -\mu_0 J_z$$

or

$$\boxed{\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}} \tag{64}$$

Returning to (60), we can now substitute for the divergence and Laplacian of **A** and obtain the desired answer,

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{28}$$

We have already shown the use of Stokes' theorem in obtaining the integral form of Ampère's circuital law from (28) and need not repeat that labor here.

We thus have succeeded in showing that every result we have essentially pulled from thin air[11] for magnetic fields follows from the basic definitions of **H**, **B**, and **A**. The derivations are not simple, but they should be understandable on a step-by-step basis.

Finally, let us return to (64) and make use of this formidable second-order vector partial differential equation to find the vector magnetic potential in one simple example. We select the field between conductors of a coaxial cable, with radii of a and b as usual, and current I in the $\mathbf{a}_z$ direction in the inner conductor. Between the conductors, $\mathbf{J} = 0$, and therefore

$$\nabla^2 \mathbf{A} = 0$$

We have already been told (and Problem 7.44 gives us the opportunity to check the results for ourselves) that the vector Laplacian may be expanded as the vector sum of the scalar Laplacians of the three components in rectangular coordinates,

$$\nabla^2 \mathbf{A} = \nabla^2 A_x \mathbf{a}_x + \nabla^2 A_y \mathbf{a}_y + \nabla^2 A_z \mathbf{a}_z$$

but such a relatively simple result is not possible in other coordinate systems. That is, in cylindrical coordinates, for example,

$$\nabla^2 \mathbf{A} \neq \nabla^2 A_\rho \mathbf{a}_\rho + \nabla^2 A_\phi \mathbf{a}_\phi + \nabla^2 A_z \mathbf{a}_z$$

However, it is not difficult to show for cylindrical coordinates that the z component of the vector Laplacian is the scalar Laplacian of the z component of **A**, or

$$\nabla^2 \mathbf{A}\Big|_z = \nabla^2 A_z \tag{65}$$

and because the current is entirely in the z direction in this problem, **A** has only a z component. Therefore,

$$\nabla^2 A_z = 0$$

or

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial A_z}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 A_z}{\partial \phi^2} + \frac{\partial^2 A_z}{\partial z^2} = 0$$

Thinking symmetrical thoughts about (51) shows us that A_z is a function only of ρ, and thus

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{dA_z}{d\rho}\right) = 0$$

We have solved this equation before, and the result is

$$A_z = C_1 \ln \rho + C_2$$

If we choose a zero reference at $\rho = b$, then

$$A_z = C_1 \ln \frac{\rho}{b}$$

[11] Free space.

In order to relate C_1 to the sources in our problem, we may take the curl of **A**,

$$\nabla \times \mathbf{A} = -\frac{\partial A_z}{\partial \rho}\mathbf{a}_\phi = -\frac{C_1}{\rho}\mathbf{a}_\phi = \mathbf{B}$$

obtain **H**,

$$\mathbf{H} = -\frac{C_1}{\mu_0 \rho}\mathbf{a}_\phi$$

and evaluate the line integral,

$$\oint \mathbf{H} \cdot d\mathbf{L} = I = \int_0^{2\pi} -\frac{C_1}{\mu_0 \rho}\mathbf{a}_\phi \cdot \rho\, d\phi\, \mathbf{a}_\phi = -\frac{2\pi C_1}{\mu_0}$$

Thus

$$C_1 = -\frac{\mu_0 I}{2\pi}$$

or

$$A_z = \frac{\mu_0 I}{2\pi} \ln \frac{b}{\rho} \tag{66}$$

and

$$H_\phi = \frac{I}{2\pi\rho}$$

as before. A plot of A_z versus ρ for $b = 5a$ is shown in Figure 7.20; the decrease of $|\mathbf{A}|$ with distance from the concentrated current source that the inner conductor represents is evident. The results of Problem D7.9 have also been added to Figure 7.20. The extension of the curve into the outer conductor is left as Problem 7.43.

It is also possible to find A_z between conductors by applying a process some of us informally call "uncurling." That is, we know **H** or **B** for the coax, and we may

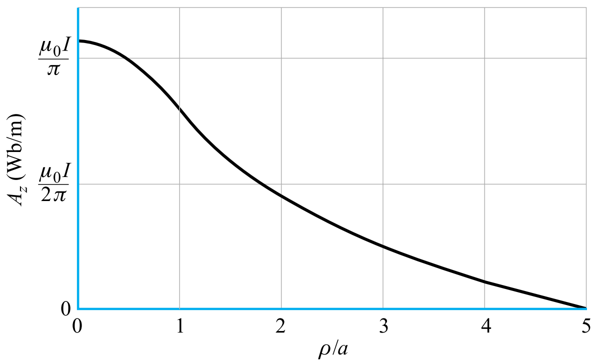

Figure 7.20 The vector magnetic potential is shown within the inner conductor and in the region between conductors for a coaxial cable with $b = 5a$ carrying I in the $\mathbf{a}_z$ direction. $A_z = 0$ is arbitrarily selected at $\rho = b$.

therefore select the ϕ component of $\nabla \times \mathbf{A} = \mathbf{B}$ and integrate to obtain A_z. Try it, you'll like it!

D7.10. Equation (66) is obviously also applicable to the exterior of any conductor of circular cross section carrying a current I in the $\mathbf{a}_z$ direction in free space. The zero reference is arbitrarily set at $\rho = b$. Now consider two conductors, each of 1 cm radius, parallel to the z axis with their axes lying in the $x = 0$ plane. One conductor whose axis is at $(0, 4\,\text{cm}, z)$ carries 12 A in the $\mathbf{a}_z$ direction; the other axis is at $(0, -4\text{ cm}, z)$ and carries 12 A in the $-\mathbf{a}_z$ direction. Each current has its zero reference for $\mathbf{A}$ located 4 cm from its axis. Find the total $\mathbf{A}$ field at: (*a*) $(0, 0, z)$; (*b*) $(0, 8\,\text{cm}, z)$; (*c*) $(4\,\text{cm}, 4\,\text{cm}, z)$; (*d*) $(2\,\text{cm}, 4\,\text{cm}, z)$.

Ans. 0; 2.64 μWb/m; 1.93 μWb/m; 3.40 μWb/m

REFERENCES

1. Boast, W. B. (See References for Chapter 2.) The scalar magnetic potential is defined on p. 220, and its use in mapping magnetic fields is discussed on p. 444.
2. Jordan, E. C., and K. G. Balmain. *Electromagnetic Waves and Radiating Systems.* 2d ed. Englewood Cliffs, N.J.: Prentice-Hall, 1968. Vector magnetic potential is discussed on pp. 90–96.
3. Paul, C. R., K. W. Whites, and S. Y. Nasar. *Introduction to Electromagnetic Fields*. 3d ed. New York: McGraw-Hill, 1998. The vector magnetic potential is presented on pp. 216–20.
4. Skilling, H. H. (See References for Chapter 3.) The "paddle wheel" is introduced on pp. 23–25.

CHAPTER 7 PROBLEMS

7.1 (*a*) Find **H** in rectangular components at $P(2, 3, 4)$ if there is a current filament on the z axis carrying 8 mA in the $\mathbf{a}_z$ direction. (*b*) Repeat if the filament is located at $x = -1$, $y = 2$. (*c*) Find **H** if both filaments are present.

7.2 A filamentary conductor is formed into an equilateral triangle with sides of length ℓ carrying current I. Find the magnetic field intensity at the center of the triangle.

7.3 Two semi-infinite filaments on the z axis lie in the regions $-\infty < z < -a$ and $a < z < \infty$. Each carries a current I in the $\mathbf{a}_z$ direction. (*a*) Calculate **H** as a function of ρ and ϕ at $z = 0$. (*b*) What value of a will cause the magnitude of **H** at $\rho = 1$, $z = 0$, to be one-half the value obtained for an infinite filament?

7.4 Two circular current loops are centered on the z axis at $z = \pm h$. Each loop has radius a and carries current I in the $\mathbf{a}_\phi$ direction. (*a*) Find **H** on the z axis over the range $-h < z < h$. Take $I = 1$ A and plot $|\mathbf{H}|$ as a function of z/a if

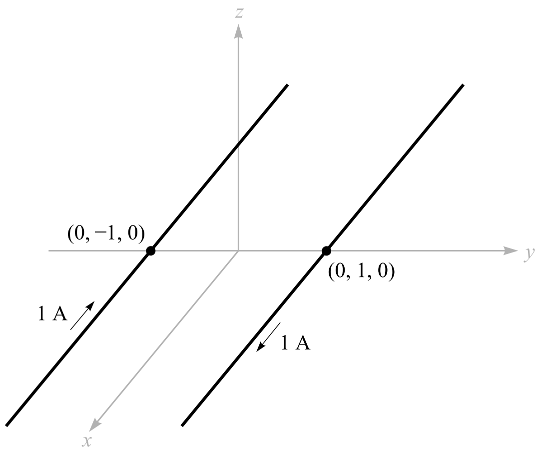

Figure 7.21 See Problem 7.5.

(b) $h = a/4$; (c) $h = a/2$; (d) $h = a$. Which choice for h gives the most uniform field? These are called Helmholtz coils (of a single turn each in this case), and are used in providing uniform fields.

7.5 The parallel filamentary conductors shown in Figure 7.21 lie in free space. Plot $|\mathbf{H}|$ versus y, $-4 < y < 4$, along the line $x = 0$, $z = 2$.

7.6 A disk of radius a lies in the xy plane, with the z axis through its center. Surface charge of uniform density ρ_s lies on the disk, which rotates about the z axis at angular velocity Ω rad/s. Find $\mathbf{H}$ at any point on the z axis.

7.7 A filamentary conductor carrying current I in the $\mathbf{a}_z$ direction extends along the entire negative z axis. At $z = 0$ it connects to a copper sheet that fills the $x > 0$, $y > 0$ quadrant of the xy plane. (a) Set up the Biot-Savart law and find $\mathbf{H}$ everywhere on the z axis; (b) repeat part (a), but with the copper sheet occupying the *entire* xy plane (Hint: express $\mathbf{a}_\phi$ in terms of $\mathbf{a}_x$ and $\mathbf{a}_y$ and angle ϕ in the integral).

7.8 For the finite-length current element on the z axis, as shown in Figure 7.5, use the Biot-Savart law to derive Eq. (9) of Section 7.1.

7.9 A current sheet $\mathbf{K} = 8\mathbf{a}_x$ A/m flows in the region $-2 < y < 2$ in the plane $z = 0$. Calculate H at $P(0, 0, 3)$.

7.10 A hollow spherical conducting shell of radius a has filamentary connections made at the top ($r = a, \theta = 0$) and bottom ($r = a, \theta = \pi$). A direct current I flows down the upper filament, down the spherical surface, and out the lower filament. Find $\mathbf{H}$ in spherical coordinates (a) inside and (b) outside the sphere.

7.11 An infinite filament on the z axis carries 20π mA in the $\mathbf{a}_z$ direction. Three $\mathbf{a}_z$-directed uniform cylindrical current sheets are also present: 400 mA/m at

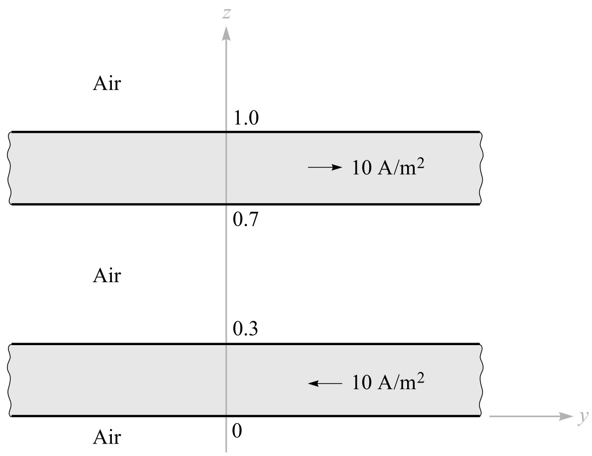

Figure 7.22 See Problem 7.12.

$\rho = 1$ cm, -250 mA/m at $\rho = 2$ cm, and -300 mA/m at $\rho = 3$ cm. Calculate H_ϕ at $\rho = 0.5$, 1.5, 2.5, and 3.5 cm.

7.12 In Figure 7.22, let the regions $0 < z < 0.3$ m and $0.7 < z < 1.0$ m be conducting slabs carrying uniform current densities of 10 A/m^2 in opposite directions as shown. Find **H** at $z =$: (*a*) -0.2; (*b*) 0.2; (*c*) 0.4; (*d*) 0.75; (*e*) 1.2 m.

7.13 A hollow cylindrical shell of radius a is centered on the z axis and carries a uniform surface current density of $K_a\mathbf{a}_\phi$. (*a*) Show that H is not a function of ϕ or z. (*b*) Show that H_ϕ and H_ρ are everywhere zero. (*c*) Show that $H_z = 0$ for $\rho > a$. (*d*) Show that $H_z = K_a$ for $\rho < a$. (*e*) A second shell, $\rho = b$, carries a current $K_b\mathbf{a}_\phi$. Find **H** everywhere.

7.14 A toroid having a cross section of rectangular shape is defined by the following surfaces: the cylinders $\rho = 2$ and $\rho = 3$ cm, and the planes $z = 1$ and $z = 2.5$ cm. The toroid carries a surface current density of $-50\mathbf{a}_z$ A/m on the surface $\rho = 3$ cm. Find **H** at the point $P(\rho, \phi, z)$: (*a*) $P_A(1.5\,\text{cm}, 0, 2\,\text{cm})$; (*b*) $P_B(2.1\,\text{cm}, 0, 2\,\text{cm})$; (*c*) $P_C(2.7\,\text{cm}, \pi/2, 2\,\text{cm})$; (*d*) $P_D(3.5\,\text{cm}, \pi/2, 2\,\text{cm})$.

7.15 Assume that there is a region with cylindrical symmetry in which the conductivity is given by $\sigma = 1.5e^{-150\rho}$ kS/m. An electric field of $30\mathbf{a}_z$ V/m is present. (*a*) Find **J**. (*b*) Find the total current crossing the surface $\rho < \rho_0$, $z = 0$, all ϕ. (*c*) Make use of Ampère's circuital law to find **H**.

7.16 A current filament carrying I in the $-\mathbf{a}_z$ direction lies along the entire positive z axis. At the origin, it connects to a conducting sheet that forms the xy plane. (*a*) Find **K** in the conducting sheet. (*b*) Use Ampere's circuital law to find **H** everywhere for $z > 0$; (*c*) find **H** for $z < 0$.

7.17 A current filament on the z axis carries a current of 7 mA in the $\mathbf{a}_z$ direction, and current sheets of 0.5 $\mathbf{a}_z$ A/m and -0.2 $\mathbf{a}_z$ A/m are located at $\rho = 1$ cm and $\rho = 0.5$ cm, respectively. Calculate $\mathbf{H}$ at: (a) $\rho = 0.5$ cm; (b) $\rho = 1.5$ cm; (c) $\rho = 4$ cm. (d) What current sheet should be located at $\rho = 4$ cm so that $\mathbf{H} = 0$ for all $\rho > 4$ cm?

7.18 A wire of 3 mm radius is made up of an inner material ($0 < \rho < 2$ mm) for which $\sigma = 10^7$ S/m, and an outer material (2 mm $< \rho <$ 3 mm) for which $\sigma = 4 \times 10^7$ S/m. If the wire carries a total current of 100 mA dc, determine $\mathbf{H}$ everywhere as a function of ρ.

7.19 In spherical coordinates, the surface of a solid conducting cone is described by $\theta = \pi/4$ and a conducting plane by $\theta = \pi/2$. Each carries a total current I. The current flows as a surface current radially inward on the plane to the vertex of the cone, and then flows radially outward throughout the cross section of the conical conductor. (a) Express the surface current density as a function of r; (b) express the volume current density inside the cone as a function of r; (c) determine $\mathbf{H}$ as a function of r and θ in the region between the cone and the plane; (d) determine $\mathbf{H}$ as a function of r and θ inside the cone.

7.20 A solid conductor of circular cross section with a radius of 5 mm has a conductivity that varies with radius. The conductor is 20 m long, and there is a potential difference of 0.1 V dc between its two ends. Within the conductor, $\mathbf{H} = 10^5 \rho^2 \mathbf{a}_\phi$ A/m. (a) Find σ as a function of ρ. (b) What is the resistance between the two ends?

7.21 A cylindrical wire of radius a is oriented with the z axis down its center line. The wire carries a nonuniform current down its length of density $\mathbf{J} = b\rho\, \mathbf{a}_z$ A/m^2 where b is a constant. (a) What total current flows in the wire? (b) Find $\mathbf{H}_{in}$ ($0 < \rho < a$), as a function of ρ; (c) find $\mathbf{H}_{out}(\rho > a)$, as a function of ρ; (d) verify your results of parts (b) and (c) by using $\nabla \times \mathbf{H} = \mathbf{J}$.

7.22 A solid cylinder of radius a and length L, where $L \gg a$, contains volume charge of uniform density ρ_0 C/m^3. The cylinder rotates about its axis (the z axis) at angular velocity Ω rad/s. (a) Determine the current density $\mathbf{J}$ as a function of position within the rotating cylinder. (b) Determine $\mathbf{H}$ on-axis by applying the results of Problem 7.6. (c) Determine the magnetic field intensity $\mathbf{H}$ inside and outside. (d) Check your result of part (c) by taking the curl of $\mathbf{H}$.

7.23 Given the field $\mathbf{H} = 20\rho^2 \mathbf{a}_\phi$ A/m: (a) Determine the current density $\mathbf{J}$. (b) Integrate $\mathbf{J}$ over the circular surface $\rho \le 1, 0 < \phi < 2\pi, z = 0$, to determine the total current passing through that surface in the $\mathbf{a}_z$ direction. (c) Find the total current once more, this time by a line integral around the circular path $\rho = 1, 0 < \phi < 2\pi, z = 0$.

7.24 Infinitely long filamentary conductors are located in the $y = 0$ plane at $x = n$ meters where $n = 0, \pm 1, \pm 2, \ldots$ Each carries 1 A in the $\mathbf{a}_z$ direction.

(a) Find $\mathbf{H}$ on the y axis. As a help,

$$\sum_{n=1}^{\infty} \frac{y}{y^2 + n^2} = \frac{\pi}{2} - \frac{1}{2y} + \frac{\pi}{e^{2\pi y} - 1}$$

(b) Compare your result of part (a) to that obtained if the filaments are replaced by a current sheet in the $y = 0$ plane that carries surface current density $\mathbf{K} = 1\mathbf{a}_z$ A/m.

7.25 When x, y, and z are positive and less than 5, a certain magnetic field intensity may be expressed as $\mathbf{H} = [x^2yz/(y+1)]\mathbf{a}_x + 3x^2z^2\mathbf{a}_y - [xyz^2/(y+1)]\mathbf{a}_z$. Find the total current in the $\mathbf{a}_x$ direction that crosses the strip $x = 2$, $1 \le y \le 4$, $3 \le z \le 4$, by a method utilizing: (a) a surface integral; (b) a closed line integral.

7.26 Consider a sphere of radius $r = 4$ centered at (0, 0, 3). Let S_1 be that portion of the spherical surface that lies above the xy plane. Find $\int_{S_1} (\nabla \times \mathbf{H}) \cdot d\mathbf{S}$ if $\mathbf{H} = 3\rho\, \mathbf{a}_\phi$ in cylindrical coordinates.

7.27 The magnetic field intensity is given in a certain region of space as $\mathbf{H} = [(x + 2y)/z^2]\mathbf{a}_y + (2/z)\mathbf{a}_z$ A/m. (a) Find $\nabla \times \mathbf{H}$. (b) Find $\mathbf{J}$. (c) Use $\mathbf{J}$ to find the total current passing through the surface $z = 4$, $1 \le x \le 2$, $3 \le z \le 5$, in the $\mathbf{a}_z$ direction. (d) Show that the same result is obtained using the other side of Stokes' theorem.

7.28 Given $\mathbf{H} = (3r^2/\sin\theta)\mathbf{a}_\theta + 54r\cos\theta\,\mathbf{a}_\phi$ A/m in free space: (a) Find the total current in the $\mathbf{a}_\theta$ direction through the conical surface $\theta = 20^\circ$, $0 \le \phi \le 2\pi$, $0 \le r \le 5$, by whatever side of Stokes' theorem you like the best. (b) Check the result by using the other side of Stokes' theorem.

7.29 A long, straight, nonmagnetic conductor of 0.2 mm radius carries a uniformly distributed current of 2 A dc. (a) Find $\mathbf{J}$ within the conductor. (b) Use Ampère's circuital law to find $\mathbf{H}$ and $\mathbf{B}$ within the conductor. (c) Show that $\nabla \times \mathbf{H} = \mathbf{J}$ within the conductor. (d) Find $\mathbf{H}$ and $\mathbf{B}$ outside the conductor. (e) Show that $\nabla \times \mathbf{H} = \mathbf{J}$ outside the conductor.

7.30 (An inversion of Problem 7.20.) A solid, nonmagnetic conductor of circular cross section has a radius of 2 mm. The conductor is inhomogeneous, with $\sigma = 10^6(1 + 10^6\rho^2)$ S/m. If the conductor is 1 m in length and has a voltage of 1 mV between its ends, find: (a) $\mathbf{H}$ inside; (b) the total magnetic flux inside the conductor.

7.31 The cylindrical shell defined by 1 cm $< \rho <$ 1.4 cm consists of a nonmagnetic conducting material and carries a total current of 50 A in the $\mathbf{a}_z$ direction. Find the total magnetic flux crossing the plane $\phi = 0$, $0 < z < 1$: (a) $0 < \rho <$ 1.2 cm; (b) 1.0 cm $< \rho <$ 1.4 cm; (c) 1.4 cm $< \rho <$ 20 cm.

7.32 The free space region defined by $1 < z < 4$ cm and $2 < \rho < 3$ cm is a toroid of rectangular cross section. Let the surface at $\rho = 3$ cm carry a surface current $\mathbf{K} = 2\mathbf{a}_z$ kA/m. (a) Specify the current densities on the surfaces at

$\rho = 2$ cm, $z = 1$ cm, and $z = 4$ cm. (b) Find **H** everywhere. (c) Calculate the total flux within the toroid.

7.33 Use an expansion in rectangular coordinates to show that the curl of the gradient of any scalar field G is identically equal to zero.

7.34 A filamentary conductor on the z axis carries a current of 16 A in the $\mathbf{a}_z$ direction, a conducting shell at $\rho = 6$ carries a total current of 12 A in the $-\mathbf{a}_z$ direction, and another shell at $\rho = 10$ carries a total current of 4 A in the $-\mathbf{a}_z$ direction. (a) Find **H** for $0 < \rho < 12$. (b) Plot H_ϕ versus ρ. (c) Find the total flux Φ crossing the surface $1 < \rho < 7$, $0 < z < 1$, at fixed ϕ.

7.35 A current sheet, $\mathbf{K} = 20\,\mathbf{a}_z$ A/m, is located at $\rho = 2$, and a second sheet, $\mathbf{K} = -10\mathbf{a}_z$ A/m, is located at $\rho = 4$. (a) Let $V_m = 0$ at $P(\rho = 3, \phi = 0, z = 5)$ and place a barrier at $\phi = \pi$. Find $V_m(\rho, \phi, z)$ for $-\pi < \phi < \pi$. (b) Let $\mathbf{A} = 0$ at P and find $\mathbf{A}(\rho, \phi, z)$ for $2 < \rho < 4$.

7.36 Let $\mathbf{A} = (3y - z)\mathbf{a}_x + 2xz\mathbf{a}_y$ Wb/m in a certain region of free space. (a) Show that $\nabla \cdot \mathbf{A} = 0$. ($b$) At $P(2, -1, 3)$, find **A**, **B**, **H**, and **J**.

7.37 Let $N = 1000$, $I = 0.8$ A, $\rho_0 = 2$ cm, and $a = 0.8$ cm for the toroid shown in Figure 7.12b. Find V_m in the interior of the toroid if $V_m = 0$ at $\rho = 2.5$ cm, $\phi = 0.3\pi$. Keep ϕ within the range $0 < \phi < 2\pi$.

7.38 A square filamentary differential current loop, dL on a side, is centered at the origin in the $z = 0$ plane in free space. The current I flows generally in the $\mathbf{a}_\phi$ direction. (a) Assuming that $r >> dL$, and following a method similar to that in Section 4.7, show that

$$d\mathbf{A} = \frac{\mu_0 I (dL)^2 \sin\theta}{4\pi r^2}\,\mathbf{a}_\phi$$

(b) Show that

$$d\mathbf{H} = \frac{I(dL)^2}{4\pi r^3}(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta)$$

The square loop is one form of a *magnetic dipole*.

7.39 Planar current sheets of $\mathbf{K} = 30\mathbf{a}_z$ A/m and $-30\mathbf{a}_z$ A/m are located in free space at $x = 0.2$ and $x = -0.2$, respectively. For the region $-0.2 < x < 0.2$ (a) find **H**; (b) obtain an expression for V_m if $V_m = 0$ at $P(0.1, 0.2, 0.3)$; (c) find **B**; (d) obtain an expression for **A** if $\mathbf{A} = 0$ at P.

7.40 Show that the line integral of the vector potential **A** about any closed path is equal to the magnetic flux enclosed by the path, or $\oint \mathbf{A} \cdot d\mathbf{L} = \int \mathbf{B} \cdot d\mathbf{S}$.

7.41 Assume that $\mathbf{A} = 50\rho^2\mathbf{a}_z$ Wb/m in a certain region of free space. (a) Find **H** and **B**. (b) Find **J**. (c) Use **J** to find the total current crossing the surface $0 \le \rho \le 1$, $0 \le \phi < 2\pi$, $z = 0$. (d) Use the value of H_ϕ at $\rho = 1$ to calculate $\oint \mathbf{H} \cdot d\mathbf{L}$ for $\rho = 1$, $z = 0$.

7.42 Show that $\nabla_2(1/R_{12}) = -\nabla_1(1/R_{12}) = \mathbf{R}_{21}/R_{12}^3$.

7.43 Compute the vector magnetic potential within the outer conductor for the coaxial line whose vector magnetic potential is shown in Figure 7.20 if the outer radius of the outer conductor is $7a$. Select the proper zero reference and sketch the results on the figure.

7.44 By expanding Eq. (58), Section 7.7 in rectangular coordinates, show that (59) is correct.